was independently discovered by Cox (Ref. 5-21) and by de Boor (Ref. 5-22). Gordon and Riesenfeld (Refs. 5-15 and 5-23) applied the B-spline basis to curve definition.

Again letting $P(t)$ be the position vectors along the curve as a function of the parameter t, a B-spline curve is given by

$$P(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t) \qquad t_{\min} \leq t < t_{\max}, \quad 2 \leq k \leq n+1 \tag{5-83}$$

where the B_i are the position vectors of the $n+1$ defining polygon vertices and the $N_{i,k}$ are the normalized B-spline basis functions.

For the ith normalized B-spline basis function of order k (degree $k-1$), the basis functions $N_{i,k}(t)$ are defined by the Cox-deBoor recursion formulas. Specifically,

$$N_{i,1}(t) = \begin{cases} 1 & \text{if } x_i \leq t < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{5-84a}$$

and

$$N_{i,k}(t) = \frac{(t - x_i)N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}} \tag{5-84b}$$

The values of x_i are elements of a knot vector satisfying the relation $x_i \leq x_{i+1}$. The parameter t varies from $t_{\min}$ to $t_{\max}$ along the curve $P(t)$.† The convention $0/0 = 0$ is adopted.

Formally a B-spline curve is defined as a polynomial spline function of order k (degree $k-1$) since it satisfies the following two conditions:

The function $P(t)$ is a polynomial of degree $k-1$ on each interval $x_i \leq t < x_{i+1}$.

$P(t)$ and its derivatives of order $1, 2, \cdots, k-2$ are all continuous over the entire curve.

Thus, for example, a fourth-order B-spline curve is a piecewise cubic curve.

Because a B-spline basis is used to describe a B-spline curve, several properties in addition to those already mentioned above are immediately known:

The sum of the B-spline basis functions for any parameter value t can be shown (see Refs. 5-15 and 5-22) to be

$$\sum_{i=1}^{n+1} N_{i,k}(t) \equiv 1 \tag{5-85}$$

Each basis function is positive or zero for all parameter values, i.e., $N_{i,k} \geq 0$.

Except for $k = 1$ each basis function has precisely one maximum value.

† Here note that, in contrast to the Bézier curve, the defining polygon vertices or points are

The maximum order of the curve is equal to the number of defining polygon vertices.

The curve exhibits the variation diminishing property. Thus the curve does not oscillate about any straight line more often than its defining polygon.

The curve generally follows the shape of the defining polygon.

Any affine transformation can be applied to the curve by applying it to the defining polygon vertices; i.e., the curve is transformed by transforming the defining polygon vertices.

The curve lies within the convex hull of its defining polygon.

In fact, the convex hull property of B-spline curves is stronger than that for Bézier curves. For a B-spline curve of order k (degree $k-1$) a point on the curve lies within the convex hull of k neighboring points. Thus, all points on a B-spline curve must lie within the union of *all* such convex hulls formed by taking k successive defining polygon vertices. Figure 5-32, where the convex hulls are shown shaded, illustrates this effect for different values of k. Notice in particular that for $k = 2$ the convex hull is just the defining polygon itself. Hence, the B-spline curve is also just the defining polygon itself.

Using the convex hull property it is easy to see that if all the defining polygon vertices are colinear, then the resulting B-spline curve is a straight line for all orders k. Further, if l colinear polygon vertices occur in a noncolinear defining polygon, then the straight portions of the defining curve (if any) start and end at least $k-2$ spans from the beginning and end of the series of colinear polygon vertices. If the series of colinear polygon vertices is completely contained within a noncolinear defining polygon, then the number of colinear curve spans is at least $l - 2k + 3$. If the series of colinear polygon vertices occurs at the end of a noncolinear defining polygon, then the number of colinear curve spans is at least $l - k + 1$. Figure 5-33 illustrates these results.

If at least $k-1$ coincident defining polygon vertices occur, i.e., $B_i = B_{i+1} \cdots = B_{i+k-2}$, then the convex hull of B_i to B_{i+k-2} is the vertex itself. Hence, the resulting B-spline curve must pass through the vertex B_i. Figure 5-34 illustrates this point for $k = 3$. Further, since a B-spline curve is everywhere C^{k-2} continuous, it is C^{k-2} continuous at B_i.

Finally, note that because of these continuity properties B-spline curves smoothly transition, with C^{k-2} continuity, into embedded straight segments as shown in Fig. 5-35.

Equations (5-84) clearly show that the choice of knot vector has a significant influence on the B-spline basis functions $N_{i,k}(t)$ and hence on the resulting B-spline curve. The only requirement for a knot vector is that it satisfy the relation $x_i \leq x_{i+1}$; i.e., it is a monotonically increasing series of real numbers. Fundamentally three types of knot vector are used: uniform, open uniform (or open) and nonuniform.

In a uniform knot vector, individual knot values are evenly spaced. Examples are for $N_{i,3}$ are given by the following diagram:

$$\begin{array}{cccccc} N_{1,3} & N_{2,3} & N_{3,3} & N_{4,3} & & \\ N_{1,2} & N_{2,2} & N_{3,2} & N_{4,2} & N_{5,2} & \\ N_{1,1} & N_{2,1} & N_{3,1} & N_{4,1} & N_{5,1} & N_{6,1} \end{array}$$

The inverse dependencies for $i \geq 1$ are given by

$$\begin{array}{cccccc} N_{1,3} & N_{2,3} & N_{3,3} & N_{4,3} & N_{5,3} & N_{6,3} \\ N_{1,2} & N_{2,2} & N_{3,2} & N_{4,2} & N_{5,2} & \\ N_{1,1} & N_{2,1} & N_{3,1} & N_{4,1} & & \end{array}$$

Now, what is the knot vector range needed for this calculation? Equation (5–84) shows that the calculation of $N_{6,1}$ requires knot values x_6 and x_7, while calculation of $N_{1,1}$ requires x_1 and x_2. Thus, knot values from 0 to $n+k$ are required. The number of knot values is thus $n+k+1$. Hence the knot vector for these periodic basis functions is

$$[\,X\,] = [\,0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6\,]$$

where $x_1 = 0, \cdots, x_7 = 6$. The parameter range is $0 \leq t \leq 6$. Using Eq. (5–84) and the dependency diagram above, the basis functions for various parameter ranges are

$0 \leq t < 1$

$$N_{1,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \neq 1$$
$$N_{1,2}(t) = t; \qquad N_{i,2}(t) = 0, \quad i \neq 1$$
$$N_{1,3}(t) = t^2; \qquad N_{i,3}(t) = 0, \quad i \neq 1$$

$1 \leq t < 2$

$$N_{2,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \neq 2$$
$$N_{1,2}(t) = (1-t); \qquad N_{2,2}(t) = (t-1); \qquad N_{i,2}(t) = 0, \quad i \neq 1,2$$
$$N_{1,3}(t) = \frac{t}{2}(1-t) + \left(\frac{3-t}{2}\right)(t-1);$$
$$N_{2,3}(t) = \frac{(t-1)^2}{2}; \qquad N_{i,3}(t) = 0, \quad i \neq 1,2,3$$

$2 \leq t < 3$

$$N_{3,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \neq 3$$
$$N_{2,2}(t) = (3-t); \qquad N_{3,2}(t) = (t-2); \qquad N_{i,2}(t) = 0, \quad i \neq 2,3$$
$$N_{1,3}(t) = \frac{(3-t)^2}{2};$$
$$N_{2,3}(t) = \frac{(t-1)(3-t)}{2} + \frac{(4-t)(t-2)}{2};$$
$$N_{3,3}(t) = \frac{(t-2)^2}{2}; \qquad N_{i,3}(t) = 0, \quad i \neq 1,2,3$$

$3 \leq t < 4$

$$N_{4,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \neq 4$$
$$N_{3,2}(t) = (4-t); \qquad N_{4,2}(t) = (t-3); \qquad N_{i,2}(t) = 0, \quad i \neq 3,4$$
$$N_{2,3}(t) = \frac{(4-t)^2}{2}; \qquad N_{3,3}(t) = \frac{(t-2)(4-t)}{2} + \frac{(5-t)(t-3)}{2};$$
$$N_{4,3}(t) = \frac{(t-3)^2}{2}; \qquad N_{i,3}(t) = 0, \quad i \neq 2,3,4$$

$4 \leq t < 5$

$$N_{5,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \neq 5$$
$$N_{4,2}(t) = (5-t); \qquad N_{5,2}(t) = (t-4); \qquad N_{i,2}(t) = 0, \quad i \neq 4,5$$
$$N_{3,3}(t) = \frac{(5-t)^2}{2};$$
$$N_{4,3}(t) = \frac{(t-3)(5-t)}{2} + \frac{(6-t)(t-4)}{2};$$
$$N_{i,3}(t) = 0, \quad i \neq 3,4$$

$5 \leq t < 6$

$$N_{6,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \neq 6$$
$$N_{5,2}(t) = (6-t); \qquad N_{i,2}(t) = 0, \quad i \neq 5$$
$$N_{4,3}(t) = \frac{(6-t)^2}{2}; \qquad N_{i,3}(t) = 0, \quad i \neq 4$$

Note that because of the $<$ sign in the definition of $N_{i,1}$, all basis functions are precisely zero at $t = 6$.

These results are shown in Fig. 5–36 and Fig. 5–39c. Note that each one of the basis functions is a piecewise parabolic (quadratic) curve. Here, three piecewise parabolic segments on the intervals $x_i \rightarrow x_{i+1}$, $x_{i+1} \rightarrow x_{i+2}$, $x_{i+2} \rightarrow x_{i+3}$ are joined together to form each $N_{i,3}$ basis function. Further, note that each of the basis functions is simply a translate of the other.

Using the results of Ex. 5–9, the buildup of the higher order basis functions $N_{i,3}$ from lower order basis functions is easily illustrated. Figure 5–39a shows the first-order basis functions determined in Ex. 5–9, Fig. 5–39b shows the second-order basis functions and Fig. 5–39c repeats the third-order basis functions of Fig. 5–36 for completeness. Notice how the range of nonzero basis function values spreads with increasing order. The basis function is said to provide support on the interval x_i to x_{i+k}.

Examining Fig. 5–36 closely reveals an important property of uniform basis functions. Recalling from Eq. 5–85 that $\sum N_{i,k}(t) = 1$ at any parameter value t shows that a complete set of periodic basis functions for $k = 3$ is defined only in the range $2 \leq t \leq 4$. Outside of this range the $\sum N_{i,k}(t) \neq 1$. For a uniform knot vector beginning at 0 with integer spacings the usable parameter range is $k-1 \leq t \leq (n+k)-(k-1) = n+1$. For more general or normalized knot vectors, the reduction in usable parameter range corresponds to the loss of $k-1$ knot value intervals at each end of the knot vector.

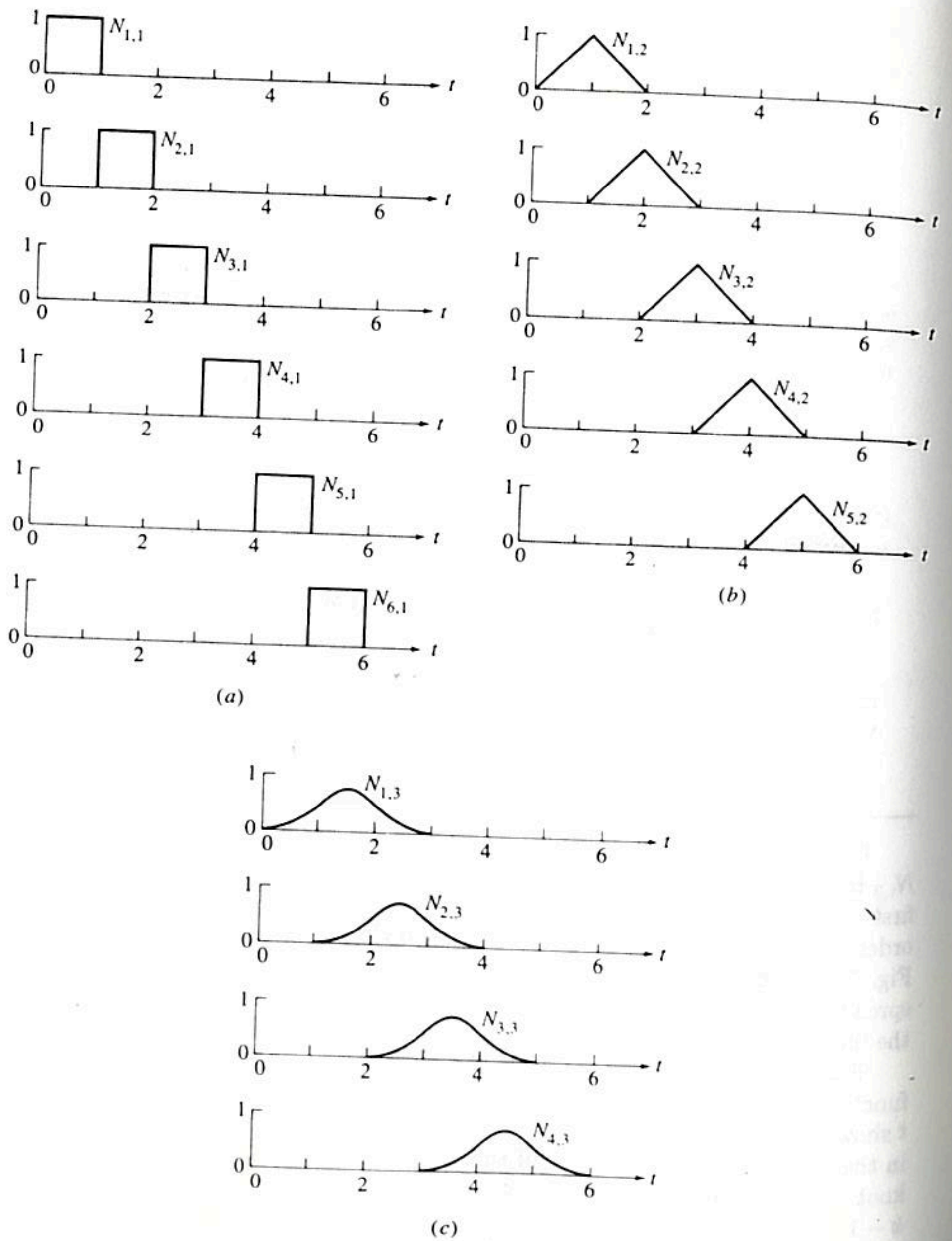

Figure 5–39 Periodic basis function buildup $n+1=4$. (a) $k=1$; (b) $k=2$; (c) $k=3$.

Example 5–10 Calculating Open Uniform Basis Functions

Calculate the four ($n=3$) third-order ($k=3$) basis functions $N_{i,3}(t)$, $i = 1,2,3,4$ with an open knot vector.

Recalling that formally an open knot vector with integer intervals between internal knot values is given by

$$x_i = 0 \qquad 1 \le i \le k$$
$$x_i = i - k \qquad k+1 \le i \le n+1$$
$$x_i = n - k + 2 \qquad n+2 \le i \le n+k+1$$

The parameter range is $0 \le t \le n-k+2$, i.e., from zero to the maximum knot value. Again, as in Ex. 5–9, the number of knot values is $n+k+1$. Using integer knot values, the knot vector for the current example is

$$[\,X\,] = [\,0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 2 \quad 2\,]$$

where $x_1 = 0, \cdots, x_7 = 2$. The parameter range is from $0 \le t \le 2$.

Using Eqs. (5–84) and the dependency diagrams, the basis functions for various parameter ranges are

$0 \le t < 1$

$$N_{3,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \ne 3$$
$$N_{2,2}(t) = 1 - t; \qquad N_{3,2}(t) = t; \qquad N_{i,2}(t) = 0, \quad i \ne 2,3$$
$$N_{1,3}(t) = (1-t)^2; \qquad N_{2,3}(t) = t(1-t) + \frac{(2-t)}{2}t;$$
$$N_{3,3}(t) = \frac{t^2}{2}; \qquad N_{i,3}(t) = 0, \quad i \ne 1,2,3$$

$1 \le t < 2$

$$N_{4,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \ne 4$$
$$N_{3,2}(t) = (2-t); \qquad N_{4,2}(t) = (t-1); \qquad N_{i,2}(t) = 0, \quad i \ne 3,4$$
$$N_{2,3}(t) = \frac{(2-t)^2}{2}; \qquad N_{3,3}(t) = \frac{t(2-t)}{2} + (2-t)(t-1);$$
$$N_{4,3}(t) = (t-1)^2; \qquad N_{i,3}(t) = 0, \quad i \ne 2,3,4$$

These results are shown in Fig. 5–40.

Comparing the results from Ex. 5–10 shown in Fig. 5–40 with those from Ex. 5–9 shown in Fig. 5–39 illustrates that significantly different results are obtained when using periodic uniform or open uniform knot vectors. In particular, note that for open uniform knot vectors a complete set of basis functions is defined for the entire parameter range; i.e., $\sum N_{i,k}(t) = 1$ for all t, $0 \le t \le n-k+2$. Notice also the reduction in parameter range compared to that for a periodic uniform knot vector.

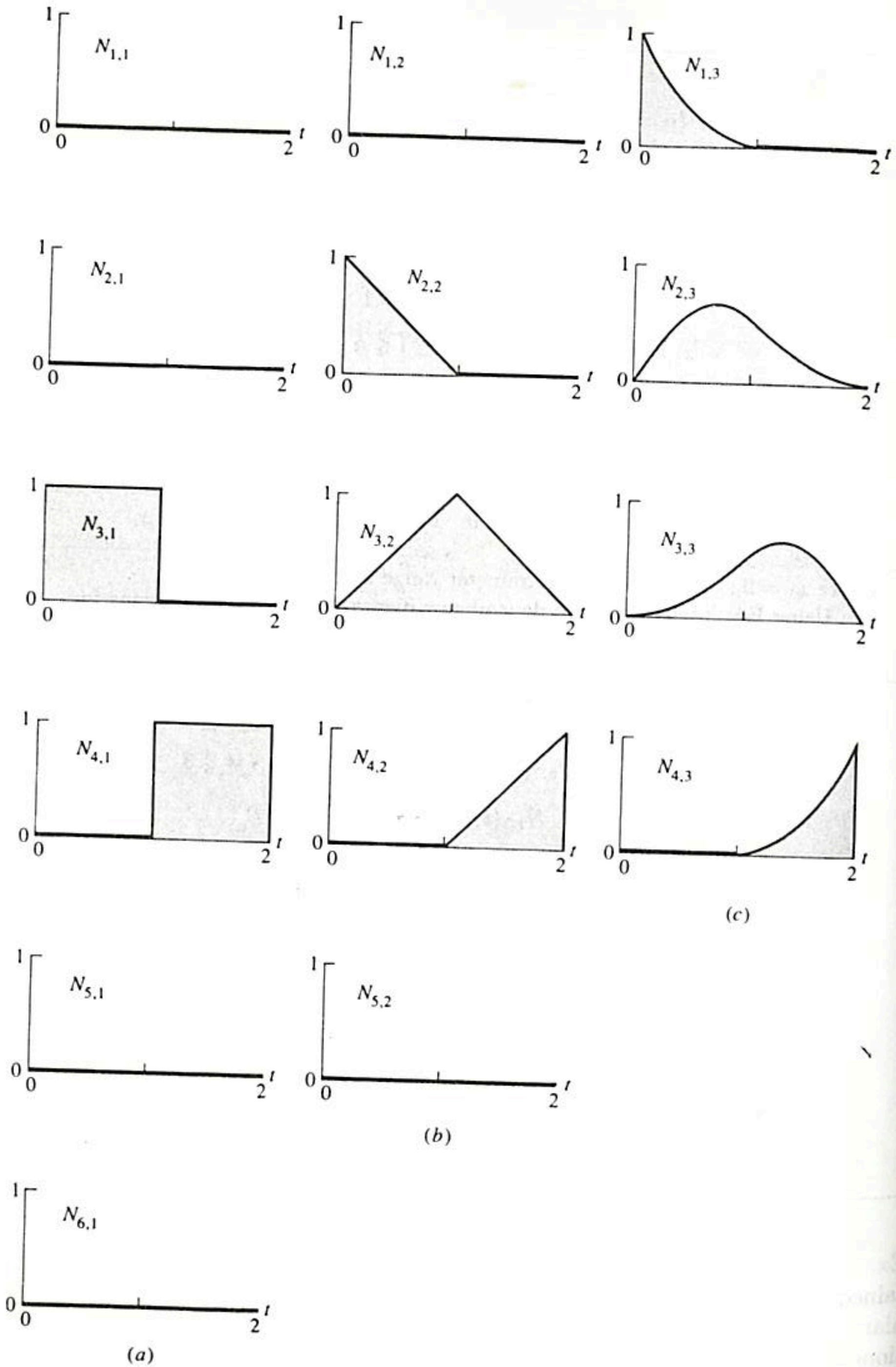

Figure 5–40 Open basis function buildup $n + 1 = 4$. (a) $k = 1$; (b) $k = 2$; (c) $k = 3$.

Example 5–11 Calculating Nonuniform Basis Functions

Calculate the five $(n + 4)$ third-order $(k = 3)$ basis functions $N_{i,3}(t)$, $i = 1, 2, 3, 4, 5$ using the knot vector $[\,X\,] = [\,0 \;\; 0 \;\; 0 \;\; 1 \;\; 1 \;\; 3 \;\; 3 \;\; 3\,]$ which contains an interior repeated knot value. Using Eqs. (5–84) and the dependency diagrams, the basis functions are

$0 \le t < 1$

$$N_{3,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \neq 2$$
$$N_{2,2}(t) = 1 - t; \qquad N_{3,2}(t) = t; \qquad N_{i,2}(t) = 0, \quad i \neq 2, 3$$
$$N_{1,3}(t) = (1 - t)^2; \qquad N_{2,3}(t) = t(1 - t) + (1 - t)t = 2t(1 - t);$$
$$N_{3,3}(t) = t^2; \qquad N_{i,3}(t) = 0, \quad i \neq 1, 2, 3$$

$1 \le t < 1$

$$N_{i,1}(t) = 0, \quad \text{all } i$$
$$N_{i,2}(t) = 0, \quad \text{all } i$$
$$N_{i,3}(t) = 0, \quad \text{all } i$$

Notice specifically that as a consequence of the multiple knot value, $N_{4,1}(t) = 0$ for all t.

$1 \le t < 3$

$$N_{5,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \neq 5$$
$$N_{4,2}(t) = \frac{(3 - t)}{2}; \qquad N_{5,2}(t) = \frac{(t - 1)}{2}; \qquad N_{i,2}(t) = 0, \quad i \neq 4, 5$$
$$N_{3,3}(t) = \frac{(3 - t)^2}{4};$$
$$N_{4,3}(t) = \frac{(t - 1)(3 - t)}{4} + \frac{(3 - t)(t - 1)}{4} = \frac{(3 - t)(t - 1)}{2};$$
$$N_{5,3}(t) = \frac{(t - 1)^2}{4}; \qquad N_{i,3}(t) = 0, \quad i \neq 3, 4, 5$$

The results are shown in Fig. 5–38d.

Notice that for each value of t the $\sum N_{i,k}(t) = 1.0$. For example, with $0 \le t < 1$

$$\sum_{i=1}^{5} N_{i,3}(t) = (1 - t)^2 + 2t(1 - t) + t^2 = 1 - 2t + t^2 + 2t - 2t^2 + t^2 = 1$$

Similarly, for $1 \le t < 3$

$$\sum_{i=1}^{5} N_{i,3}(t) = \frac{1}{4}[(3 - t)^2 + 2(3 - t)(t - 1) + (t - 1)^2]$$
$$= \frac{1}{4}[9 - 6t + t^2 - 6 + 8t - 2t^2 + 1 - 2t + t^2]$$
$$= \frac{4}{4} = 1$$