An approach to drawing fair plane open curves*

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Abstract

In this paper an approach for interpolating a given sequence of points by a *fair* plane curve is presented. Since the fairness concept is subjective, a non-classical modeling tool - fuzzy sets - is used and interactive facilities are provided. Some results of our implementation are included.

1 Introduction

One of the old geometrical problems that have challenged researchers is to find out a fair curve that interpolates a given sequence of points. Some classical methods, such as Lagrange's and Newton's interpolation may tend to oscillate more and more between a pair of interpolated points, as the degree of the used polynomials increases [1]. To avoid such kind of numerical instability, Schoenberg introduced in 1946 the technique of spline interpolation [2, 3]. Instead of approximating a curve by a single polynomial of high degree, this technique uses a set of polynomial pieces of lower degree called spline functions. In this way, the graph of a function may be numerically stable. Among a great variety of spline functions, those ones whose parameters have geometrical meanings, such as cardinal splines, B-splines, Bézier splines [4] and Beta splines [5], have gained popularity among geometrical designers. However, from the designer's point of view, the resulting curve may still not be enough fair. Therefore, additional conditions should be established.

Although the definition of fairness is subjective there are some attempts to give quantitative measure of fairness. According to Su and Liu [6] a plane curve is called fair if the following three conditions are satisfied:

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- the curve has GC^2-continuity (concept of smoothness);
- there are no unwanted inflection-points on the curve; and
- its curvature varies in an even manner.

In practice, the proposed solutions for constructing a fair curve from a set of points only minimize the number of curvature extrema. What they consider as fairness means in fact smoothness. The subjective concept of "unwanted" is understood as no inflection point and "even" as curvature varying almost linearly between two subsequent points. The solutions are divided into two categories: the ones that modify simultaneously the whole data points and the ones that modify some of them. Least-square, energy and bounce method are examples of methods that involve all the data points, while cardinal spline and discrete curvature methods can be regarded as point-choosing ones. Both classes of solutions have been sucessfully applied in car, aircraft and shipbuilding industries [6].

However, we argue whether a fair shape implies necessarily that the resulting curve must not be undulant, since the notion of fairness is extremely subjective and imprecise. Suppose a set of points with a zigzag distribution is given and that the designer desires to have a *fair* curve passing through them. It would be odd to generate a curve without this zigzag shape, which the existing techniques would do, since they do not provide any mean for the designer to adapt the pre-defined objective-function regarding to his requirement. This leads us to look for a new technique that can generate, from the designer's point of view, a fair shape passing through a given set of points, no matter how odd and ambiguous is his concept of fairness.

Formally our interpolation problem can be stated as follows:

"Given a set of n points and the "grade of fairness" of the plane curve that passes through them $((x_0, sharper), (x_1, smoother), ..., (x_n, sharper))$, find an interpolatory plane open curve."

A proposal to solve this problem is divided in two sub-problems:

- the first estimation of the curve shape as a function of the given data; and
- the fine interactive adjustments of the curve shape to conform to the designer's intuitive fairness requirement.

For the first estimation it is interesting to use a curve representation that includes the curvature as its parameter, since curvature reflects directly the "smoothness behaviour" of a curve at a point. One way to implement the fine interactive adjustments of the curve shape in an ambiguous fashion seems to be to apply the techniques provided by fuzzy sets. Finally, to ensure a perfect matching of these two parts it must be decided which parameters of the curve representation should be fuzzyfied to produce more intuitive effects.

This paper is organized as follows. Section 2 presents some basic concepts necessary for understanding our method. Section 3 gives a solution for the interpolation sub-problem. Section 4 discusses a fuzzy fashion for handling the interactive adjustment sub-problem. Section 5 describes our implementation approach. Sections 6 shows some obtained results and Section 7 draws some conclusions.

2 Basic concepts

This section briefly presents some fundamentals about intrinsic geometrical properties of a curve and the use of fuzzy sets to represent inexact concepts.

2.1 Geometrical properties

In this section the useful intrinsic geometrical concepts for understanding the paper are summarized [7].

- Curvature: expresses how much the curve "bends". Formally, let $\alpha : I \rightarrow R^2$ be a curve parametrized by the arc length $s \in I$. The number $\mid \frac{d\alpha(s)}{ds} \mid = k(s)$ is called the curvature of α at s.

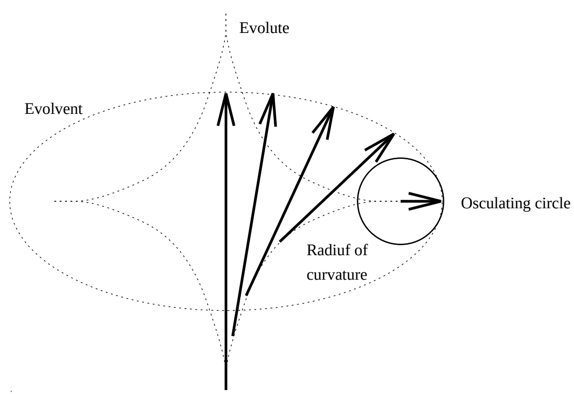

Figure 1: Intrinsic geometrical caracteristics of a curve

- Radius of curvature: is the inverse $R = \frac{1}{k}$ of the curvature (Figure 1).

- Osculating circle: is a second degree approximation of a curve, as the tangent is a first degree approximation. Formally, let $\alpha : I \rightarrow R^2$ be a curve parametrized by the arc length s, with curvature $k(s) \neq 0$, $s \in I$. The limit position of the circle passing through $\alpha(s)$, $\alpha(s + h_1)$, $\alpha(s + h_2)$ when $h_1, h_2 \rightarrow 0$ is the osculating circle at s, the center of which is on the line that supports the normal vector $n(s)$ and the radius of which is the radius of curvature $\frac{1}{k(s)}$ (Figure 1).

- Evolute: is the geometrical loci of the osculating circles centers. Formally, let $\alpha : I \rightarrow R^2$ be a regular parametrized plane curve (arbitrary parameter t), and define normal vector $n = n(t)$ and curvature $k = k(t)$. Assume that $k(t) \neq 0$, $t \in I$. In this situation, the curve $\beta(t) = \alpha(t) + \frac{1}{k(t)} n(t), t \in I$, is called the evolute of α. The curve α is called the evolvent of β (Figure 1). Pogorelov [8] states that it is possible to obtain the evolvent α from its evolute β along with one point of the evolvent. This may be done as stated (Figure 1):

1. the evolute tangents are evolvent normals. So, given an evolute, a family of evolvents is determined; and
2. given a point of the evolvent, we determine within a family which is the desired evolvent.

2.2 Fuzzy sets

Fuzzyness is a type of imprecision inherent to certain classes which do not have defined boundaries. These classes, the fuzzy sets, arise when we look for describing ambiguity, vagueness and ambivalence in mathematical models of empirical phenomena. In particular, the computer simulations of systems of high cardinality, so usual in real world, require some special non-classical mathematical formulation to deal with the imprecise descriptions. Fuzzy sets, which are classes that admits the possibility of partial membership in them, seem to be an adequate tool for dealing with such kind of problems [9].

Let X denote a space of objects. A fuzzy set A in X is a set of ordered pairs $A = \{(x, \chi_A(x)) | x \in X \text{ and } \chi_A(x) \in [0,1]\}$, with $\chi_A(x)$ being the "grade of membership of x in A". In this work we assume, for simplicity, as in [10], that $\chi_A(x)$ is a number in the interval $[0,1]$, instead of considering its values varying through a more generic algebraic structure [9, 11]. Hence, questions like $x \in X$ may have answers different from yes ($\chi_A(x) = 1$, that is, fullmembership of x) or no ($\chi_A(x) = 0$, that is, nonmembership of x).

The operations OR ($\vee$), AND ($\wedge$) and NOT ($\neg$) between fuzzy subsets A and B on X may be defined in many ways. We adopt the following definitions for these operations [9]:

- OR: $A \cup B = \{(x, Max(\chi_A(x), \chi_B(x))) | x \in X\}$;
- AND: $A \cap B = \{(x, Min(\chi_A(x), \chi_B(x))) | x \in X\}$; and
- NOT: $\neg A = \{(x, 1 - \chi_A(x)) | x \in X\}$.

As an example, one could define the concept *big radius of curvature* as a fuzzy set A. The following classifications of radii of curvature x were to be assumed:

- radii of curvature in the range of 0 to 100 are not big, so they have a null membership to *big radius of curvature* concept ($\chi_A(x) = 0, 0 \leq x < 100$);
- radii of curvature from 100 to 200 are more or less big, so they have a membership to *big radius of curvature* concept that varies linearly from null to unit ($\chi_A(x) = \frac{x-100}{100}, 100 \leq x < 200$); and
- radii of curvature above 200 are big, so they have a unitary membership to *big radius of curvature* concept ($\chi_A(x) = 1, x \geq 200$).

Figure 2 shows the fuzzy set that describes the *big radius of curvature* concept.

We observe that the grade of membership $\chi_A(x)$ of an object x in A can be interpreted as the degree of compatibility of the predicate associated with A and the object x. It is also possible to interpret $\chi_A(x)$ as the degree of possibility of x being the value of a parameter fuzzyly restricted to A.

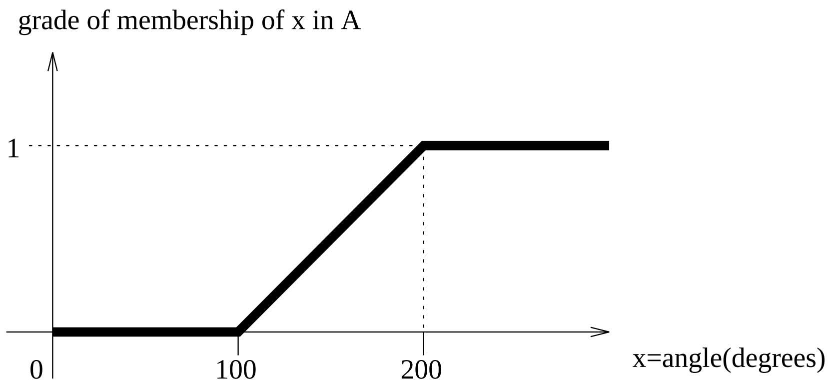

Figure 2: Big radius of curvature concept represented as a fuzzy set

3 Interpolation problem

Let S be the designer given point sequence together with the specification of "how fair" the plane curve that passes through them (grade of "local fairness") should be. The desired curve C to interpolate the points in S is looked for.

The grade of local fairness could be measured by the use of radius of curvature. Greater radii imply smoother shape at a point and lower radii, sharper shape. Furthermore, it is easier to adjust the curve shape by simply modifying locally the radius of curvature as shown in Section 4. However, it would still be necessary to find a way to model the global fairness concept of the designer. As the evolute represents the relationship among the radii of curvature, it would be chosen to model the global fairness concept of the designer!

According to Section 2, if the equation of a curve is given, then the evolute of the curve can be determined. Conversely, having the evolute and one point of a curve, one is able to detemine the curve. But the evolute of a curve is only known when the curve is known. So, what should be done to get a fair curve C having been given only some of its points and a fuzzy local fairness behavior of the curve?

Our approach is to construct the curve C from an estimated evolute, namely pseudo-evolute (PE), instead of the evolute itself. The estimation of PE is based on the designer given grade of local fairness. From PE the curve C is obtained by determining the radii-vectors, which are represented by the pair (magnitude,direction). The radii-vectors have the same direction of the PE tangents as the radii of curvature of a curve have the direction of its evolute tangents. So, to obtain C from PE is geometrically analogous to obtain any curve from its evolute. Notice that PE supports the curve global fairness concept, while the radius-vector expresses the curve local fairness concept.

In Section 5.1 an approach to the determination of directions and magnitudes of radii-vectors is explained.

4 Parameter fuzzyfication

In our case, as we reduce our problem stated in Section 1 to the construction of a fair curve from PE, which is estimated from the radii-vectors of points in S, we choose these radii-vectors as the parameter to be fuzzyfied.

The magnitude of each radius-vector is computed from a designer given SMOOTHER / SHARPER relationship related to its preceding radius-vector. This means that only the relative radius-vector values matter. Since the relationships between the magnitudes of the radii-vectors are specified by the designer (to convey his expectation of relative fairness behaviour) at the points of S, only their directions can vary freely. In order to reduce the search-space for "good" radius-vector directions, we fuzzyfy the *fair* concept by assigning to these directions a grade of membership.

For assigning a grade of membership to the chosen radius-vector directions we have established the so-called "danger zones" to avoid certain non-fairness behaviours. These danger zones allow us to formalize and measure effectively the designer concept of fairness. From them it is possible to devise the range of "values" that the radius-vector directions must not or should assume. Expressing in terms of grade of membership, we say that the values of the radius-vector directions within the danger zones have null or minimal grade of membership to the *fair* concept. The choice of a "good" radius-vector from the defined fuzzy set can be performed by a defuzzyfication procedure as explained in details in [12].

An approach for the fuzzyfication is given in Section 5.2.

5 Implementation

5.1 Interpolation solution

The main idea is to consider C as a set of pieces C_i and to use the evolvent construction to determine each of these pieces. Initially a radius-vector is estimated and associated with each point of S. With these radii-vectors and the points in S, PE is estimated and C determined from the radii-vectors. The radius-vector directions are given by the PE tangents and their magnitudes are calculated from the linear interpolation of two subsequent estimated radii-vectors for the points in S.

PE is represented by a set of Bézier curves. We decided for Bézier curves, because some geometrical caracteristics of these curves are easy to be controled, and for cubic ones, for the fact that they provide some grades of freedom in the manipulation without additional complexities.

Based on the N points of S the $N-1$ Bézier polygons B_i, $1 \leq i \leq N-1$, are computed, each one associated to a piece C_i of the curve C. In order to get a "reasonable" first estimation of PE, the polygon B_i should satisfy the two requirements [12].

- Its endpoint positions and tangents should coincide with the radius-vector magnitudes and directions of the C_i end-points.

- Let P_a, P_o, P_b be three consecutive points in S; C_a, C_b be the pieces of C, respectively, between points P_a,P_o and P_o,P_b; R_a, R_o, R_b be the radii-vectors associated, respectively, to P_a, P_o, P_b; and B_1, B_2, B_3 and B_4 the four control points of the Bézier polygon associated to C_a. These control points must satisfy:

  - $B_i = P_o + \alpha_i \overrightarrow{R_o}, \alpha_i \neq 0, i = 1, 2$; and
  - $B_i = P_a + \alpha_i \overrightarrow{R_a}, \alpha_i \neq 0, i = 3, 4$.

  The values of α_i are chosen in such a way that the convexity of the Bézier polygon is ensured.

5.2 Adjustment solution

For the implementation of radius-vector directions it is used a radius-vector angles relative to a reference line, which is ranged from 0 to 180 degrees. Without loss of generality this reference line is the x-axis of the adopted coordinate system. The range [0,180] may be used, instead of [0,360], because only the direction of the radius-vector is necessary in the determination of Bézier polygon, for each C_i. The orientation of the radii-vectors is irrelevant.

The danger zones, within which a radius-vector angle should not be, are determined from some geometrical properties that the cubic Bézier curve should satisfy as [12]:

- non-colinear condition: $\overrightarrow{R_o} \neq \beta_i(P_o - P_i)$,$\beta_i \neq 0$, $i = a, b$. If this condition fails, the Bézier polygon will degenerate to a straight line;

- non-parallel condition: $\overrightarrow{R_o} \neq \gamma_i(\overrightarrow{R_i})$,$\gamma_i \neq 0$, $i = a, b$. If this condition fails, the Bézier polygon will have: $\mid B_1 - B_2 \mid \ll \mid B_2 - B_3 \mid$ and/or $\mid B_3 - B_4 \mid \ll \mid B_2 - B_3 \mid$, causing oscilations in C_i. If oscilations are desired, this condition can be exploited to get desirable shapes; and

- non-intersection condition: $\overrightarrow{R_o} \neq \delta_a(P_o - P_a) + \epsilon_b(\overrightarrow{R_b})$,$\overrightarrow{R_o} \neq \delta_b(P_o - P_b) + \epsilon_a(\overrightarrow{R_a})$,$\delta_i$,$\epsilon_i > 0, i = a, b$. This condition controls concavity changes and oscilations. It is indeed a combination of non-parallel and non-colinear conditions.

Given a relationship GREATER/LOWER among R_a, R_o and R_b, which corresponds to the SMOOTHER/SHARPER relationship given by the designer, two situations can occur:

- the line connecting the intersection point I_1 between radii-vectors R_a and R_o and the intersection point I_2 between radii-vectors R_o and R_b crosses the convex hull of S an odd number of times: it is desired when inflection points and changes in concavity should be avoided; and

- the line connecting the intersection point I_1 between radii-vectors R_a and R_o and the intersection point I_2 between radii-vectors R_o and R_b crosses the convex hull of S an even number of times: it is desired when looking for inflection points and changes in concavity.

This analysis about the intersection points supports the identification, for each radius-vector, of its danger zones, from which the fuzzy sets on radius-vector angles are determined. Our deffuzyfication method is based on the operations on these fuzzy sets.

6 Results

Figure 3 shows a curve passing through three given points (marked with cross), its corresponding estimated evolute PE (two cubic Bézier curves) and the radii-vectors of the given interpolated points (drawn in red line). The given interpolated points, the radius-vector angles and α_i are given in the following table.

(x,y)	Radius-vector angle (rad)	α_1	α_2	α_3	α_4
(550,450)	0.35	0.50	0.25	0.50	0.25
(200,150)	1.17	0.50	0.25	0.50	0.25
(100,440)	1.30	0.50	0.25	0.50	0.25

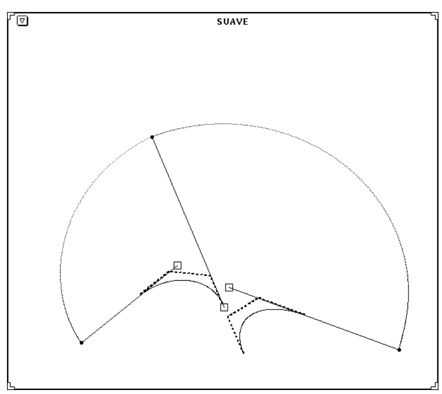

Figure 3: A fair curve passing through three points.

For the sake of clarity, the estimated evolutes PE and the radii-vectors will be omitted in the subsequent figures.

Figure 4 shows a curve obtained from six points. Notice the "zigzag" distribution of these points and the inflection points that our algorithm introduced to yield the "best" shape, regarding to our *fair* concept.

Figure 5 interpolates the same points of Figure 4. However, the shape desired in Figure 5 was one with more undulations.

Observe that different *fair* concepts lead to different "the best" fair curves.

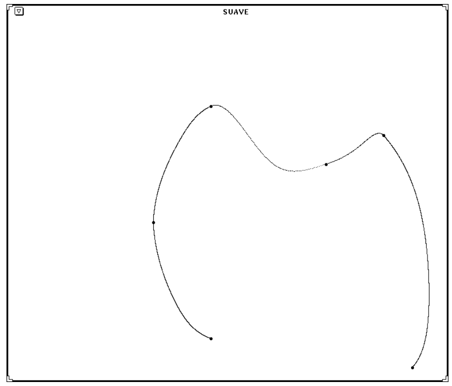

Figure 4: A fair curve with the fair concept that undulations are undesirable.

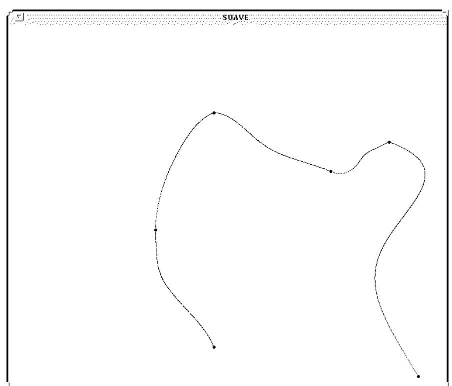

Figure 5: A fair curve with the fair concept that undulations are allowed.

7 Conclusions

An approach for the interpolation of a given sequence of points is shown. Differing from the classical approaches, our method presents two new flavours:

- it tries to capture "exactly" the *fair* concept of the designer by using non-classical foundations; and
- it is interactive by using easily manipulatable geometrical entities, so the designer may feedback his fine adjustments on the successively generated curves until the desired curve shape is achieved.

However, some questions are still open and we intend to work on them further:

- using the quadratic Bézier curves instead of cubic ones may be a good choice to PE;
- extending the method to the closed curves may be useful;
- fuzzyfying the α_i parameters may be a good way to model the designer way of changing the radius-vector magnitudes locally;

- improving the method to make better the first estimation of the radius-vector directions may lead to a faster convergence to the desired curve; and
- making a comparison between our method and the classical ones may give a better evaluation of our results.

8 Aknowledgments

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