Technical Report
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DeSMo – Deformable Surface Modeling

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Abstract

Aiming at a more realistic cloth animation, our research group has been investigated the theory of a Cosserat surface since 2005. The motivation for this technical report is to present a comprehensive treatment of this class of surfaces. The main topic of this report is to detail all the mathematical properties of the theory of a Cosserat surface presented by Green et al., with help of the material provided by Flügge. We introduce the basic elements of Differential Geometry and propose a novel way to estimate them for samples configured in an arbitrary topology. Along the exposition, we give, in a conventional notation, complete proofs for the correctness of the expressions that are relevant to our implementation. We also show how we apply the Cosserat surface to model deformable surfaces. This results in a set of differential equations, whose solution should be a series of deformable meshes. Finally, we sketch a numerical method for solving them.

Contents

1 Introduction

Aiming at realistic visual effects and an intuitive interface for cloth simulation, we investigate a novel formulation for the internal cloth forces. Our model is based on the theory of a Cosserat surface which considers the Gauss-Weingarten compatibility equations. This ensures that the regularity of surface is preserved along deformations and, thus, Differently from the works of our knowledge, unrealistic in-plane oscillations may be avoided without resorting to the intrinsic material damping terms. Furthermore, our formulation provides direct relations between the fabric statical properties, namely membrane and bending strains, and easily interpretable geometrical quantities, such as metric and curvature tensors. The main barrier for the practical applications of the theory of a Cosserat surface is its numerical complexities on meshes of artibrary topology. The main focus of our research is to derive from it expressions that are suitable to the known cloth simulation explicit integration scheme.

This document provides an overview of our on-going project on deformable surface modeling. To be self-contained, a brief overview of metric and curvature tensors are provided in Section 2. In Section 2.5 an algorithm for computing such geometric quantities from a polygonal mesh of arbitrary mesh is given. Efforts towards to achieving realistic visual fabrics behavior are summarized in Section 3. Inspired by the Theory of Cosserat Surface, Melo presented in this Doutorado's thesis a fabrics model that can produce out-of-plane movements even when in-plane forces are solely applied [12]. Before we introduce Melo and Wu's proposal in Section 6, we give a brief description of the Theory of Cosserat in Section 4. To validate their proposal, Melo and Wu adopted the semi-implicit integration schema used by Terzopoulos et al. [15]. Later, Monteiro analyzed in his Mestrado's thesis several numerical explicit integration schema for simulating cloth behaviors and showed that, in comparison with the semi-implicit schema, explicit schemata are superior both in robustness and performance [13]. Nevertheless, he still used rectilinear grids for spatial discretization; thus, his simulation solutions are limited to rectangular meshes. This is because that he kept the forward, backward and central finite difference pattern to compute the differential geometric variables that appear in the model. In order to extend our simulation domain, we present in Section 5.3 an alternative way to compute each variable in the model proposed by Melo and Wu.

2 Preliminaries

Although Archimedes of Syracuse and Apollonius of Perga have already some notions of describing the position of a point in a plane by a sequence of numbers and manipulating it with algebraic tools, only in the century XVI René Descartes introduced two fixed perpendicular lines and used the distances to them for specifying the position of a point. The distances and the fixed perpendicular lines are known as *coordinates* and *coordinate system*, respectively. Since then many coordinate systems have been developed for representing geometric and physical entities. The numbers that describe them are however not invariant under change of coordinate system. This leads to one question: how these quantities behave when passing from one coordinate system to another. In this section we present some basic ideas regarding to this issue.

2.1 Covariance and Contravariance

Knowing that the physical quantities are coordinate system invariant, their components must co-vary or contra-vary with a change of basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ to compensate the variations. When they must be contra-varied, we say that they are *contravariant vectors.* That is, the components must vary in the opposite "direction" (the inverse transformation) as the change of basis. The components of a contravariant vector are indicated by a superscript. Examples of contravariant vectors include the position vector $\mathbf{r}$ of a point

$$\mathbf{r} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

If we pass to the basis $(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3)$, the vector $\mathbf{r}$ itself does not change under this change

$$\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \mathbf{r} = \begin{bmatrix} \tilde{\mathbf{x}}_1 & \tilde{\mathbf{x}}_2 & \tilde{\mathbf{x}}_3 \end{bmatrix} \begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{bmatrix}; \tag{1}$$

instead, the components of the vector make a change that cancels the change in the reference system. Let's consider the transformation matrix T, such that

$$\begin{bmatrix} \tilde{\mathbf{x}}_1 & \tilde{\mathbf{x}}_2 & \tilde{\mathbf{x}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} T,$$

then for satisfying Eq. 1, we should have

$$\begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{bmatrix} = T^{-1} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

This implies that the components (x^1, x^2, x^3) contra-vary (transform inversely) with respect to a change of the basis $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

On the contrary, if the components of a vector should co-vary with a change of basis to maintain the same meaning, that is, they must vary by the same transformation as the change of basis, it is said to be *covariant vector*. The components of this vector are identified by a subscript. Examples of covariant vectors generally appear when taking a gradient of a function f

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x^1} & \frac{\partial f}{\partial x^2} & \frac{\partial f}{\partial x^3} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}.$$

If we pass the coordinates (x^1, x^2, x^3) to $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ via a transformation matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix},$$

that is

$$\begin{aligned} x^1 &= t_{11}\tilde{x}^1 + t_{12}\tilde{x}^2 + t_{13}\tilde{x}^3 \\ x^2 &= t_{21}\tilde{x}^1 + t_{22}\tilde{x}^2 + t_{23}\tilde{x}^3 \\ x^3 &= t_{31}\tilde{x}^1 + t_{32}\tilde{x}^2 + t_{33}\tilde{x}^3, \end{aligned}$$

the new components of ∇f may be obtained using the chain rule

$$\begin{aligned} \tilde{f}_1 &= \frac{\partial f}{\partial \tilde{x}^1} = \frac{\partial f}{\partial x^1}\frac{\partial x^1}{\partial \tilde{x}^1} + \frac{\partial f}{\partial x^2}\frac{\partial x^2}{\partial \tilde{x}^1} + \frac{\partial f}{\partial x^3}\frac{\partial x^3}{\partial \tilde{x}^1} = \frac{\partial f}{\partial x^1}t_{11} + \frac{\partial f}{\partial x^2}t_{21} + \frac{\partial f}{\partial x^3}t_{31} \\ \tilde{f}_2 &= \frac{\partial f}{\partial \tilde{x}^2} = \frac{\partial f}{\partial x^1}\frac{\partial x^1}{\partial \tilde{x}^2} + \frac{\partial f}{\partial x^2}\frac{\partial x^2}{\partial \tilde{x}^2} + \frac{\partial f}{\partial x^3}\frac{\partial x^3}{\partial \tilde{x}^2} = \frac{\partial f}{\partial x^1}t_{12} + \frac{\partial f}{\partial x^2}t_{22} + \frac{\partial f}{\partial x^3}t_{32} \\ \tilde{f}_3 &= \frac{\partial f}{\partial \tilde{x}^3} = \frac{\partial f}{\partial x^1}\frac{\partial x^1}{\partial \tilde{x}^3} + \frac{\partial f}{\partial x^2}\frac{\partial x^2}{\partial \tilde{x}^3} + \frac{\partial f}{\partial x^3}\frac{\partial x^3}{\partial \tilde{x}^3} = \frac{\partial f}{\partial x^1}t_{13} + \frac{\partial f}{\partial x^2}t_{23} + \frac{\partial f}{\partial x^3}t_{33}. \end{aligned}$$

Organizing them into matrices it is easy to see that the gradient co-varies (transforms) with a change of basis:

$$\nabla \tilde{f} = \begin{bmatrix} \tilde{f}_1 & \tilde{f}_2 & \tilde{f}_3 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} T.$$

How the coordinates of a point vary in an exactly compensating way depends, nevertheless, on the manner that they are defined. For the same vector $\mathbf{v}$ in $\Re^2$ its coordinates may be taken to be either the parallel projections or the orthogonal projections of $\mathbf{v}$ on the reference axes $\mathbf{x}_1$ and $\mathbf{x}_2$.

Taking the parallel projections, the $\mathbf{v}$ may be uniquely defined by a ordered pair (x^1, x^2) in the form

$$\mathbf{v} = x^1\mathbf{x}_1 + x^2\mathbf{x}_2 = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}. \quad (2)$$

The quantities x^1 and x^2 are the lengths of the respective parallel projections of $\mathbf{v}$ on the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ (Figure 1.(a)). We remark that these lengths are measured in units of $\sqrt{\mathbf{x}_1 \cdot \mathbf{x}_1}$ and $\sqrt{\mathbf{x}_2 \cdot \mathbf{x}_2}$. When we change the basis from $(\mathbf{x}^1, \mathbf{x}^2)$ to $(\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2)$, the vector $\mathbf{v}$ should maintain the same meaning

$$\mathbf{v} = \begin{bmatrix} \tilde{\mathbf{x}}_1 & \tilde{\mathbf{x}}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} T \begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix} = T^{-1} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}.$$

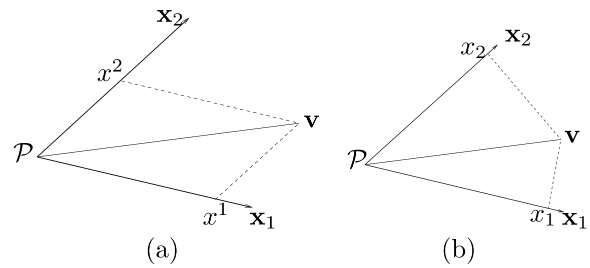

Figure 1: Components of a vector: (a) contravariant and (b) covariant components.

The quantities x^1 and x^2 vary in the opposite direction as the change of basis. Thus, they are called *contravariant components* of $\mathbf{v}$.

The vector $\mathbf{v}$ may also be uniquely described by its orthogonal projections on the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, as illustrated in Figure 1.(b)

$$\mathbf{v} = x_1\mathbf{x}_1 + x_2\mathbf{x}_2, \tag{3}$$

such that $x_1 = \mathbf{v}\cdot\mathbf{x}_1$ and $x_2 = \mathbf{v}\cdot\mathbf{x}_2$. In matricial notation, we have

$$\left[\begin{array}{cc} x_1 & x_2 \end{array}\right] = \left[\begin{array}{cc} \mathbf{v}\cdot\mathbf{x}_1 & \mathbf{v}\cdot\mathbf{x}_2 \end{array}\right] = \mathbf{v}\cdot\left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}\right].$$

The quantities x_1 and x_2 are, in fact, the lengths of the respective orthogonal projections of $\mathbf{v}$ on the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, which are measured in units of $\frac{1}{\sqrt{\mathbf{x}_1\cdot\mathbf{x}_1}}$ and $\frac{1}{\sqrt{\mathbf{x}_2\cdot\mathbf{x}_2}}$.

When we introduce new reference axes $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$, such that

$$\left[\begin{array}{cc} \tilde{\mathbf{x}}_1 & \tilde{\mathbf{x}}_2 \end{array}\right] = \left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}\right] T,$$

the coordinates $\tilde{x}_1$ and $\tilde{x}_2$ should be varied by the same transformation T to keep the vector $\mathbf{v}$ invariant. This is because that

$$\left[\begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array}\right] = \mathbf{v}\left[\begin{array}{cc} \tilde{\mathbf{x}}_1 & \tilde{\mathbf{x}}_2 \end{array}\right] = \mathbf{v}\left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}\right] T,$$

so we have

$$\left[\begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array}\right] T^{-1} = \mathbf{v}\left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}\right] = \left[\begin{array}{cc} x_1 & x_2 \end{array}\right] \Longrightarrow \left[\begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array}\right] = \left[\begin{array}{cc} x_1 & x_2 \end{array}\right] T.$$

For this reason, x_1 and x_2 are called *covariant components* of the vector $\mathbf{v}$.

The covariant and contravariant components of a vector are related with each other. Before presenting these relations, let's introduce the concept of *reciprocal basis*. For every basis $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n,)$ there exists a unique reciprocal basis $(\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^n)$, such that

$$\mathbf{x}_j\cdot\mathbf{x}^i = \delta^i_j = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases} \tag{4}$$

That is, each reciprocal basis vector $\mathbf{x}^i$ is orthogonal to all basis vectors but to one with the same index i, and the dot product of the vectors with the same index, $\mathbf{x}^i\cdot\mathbf{x}_i$, is equal to 1 (Figure 2). Moreover, the reciprocal basis relative to a given basis exists and is unique.

The component x_i of $\mathbf{v}$ in the reciprocal basis may be obtained by applying the dot product of the vector $\mathbf{v}$ with a basis vector $\mathbf{x}_i$:

$$\mathbf{v}\cdot\mathbf{x}_i = (x_1\mathbf{x}^1 + x_2\mathbf{x}^2 + \cdots + x_n\mathbf{x}^n)\cdot\mathbf{x}_i = x_i. \tag{5}$$

Conversely, the orthogonal projection of the vector $\mathbf{v}$ on a reciprocal vector $\mathbf{x}^i$ is the component x^i of $\mathbf{v}$ in the original basis

$$\mathbf{v}\cdot\mathbf{x}^i = (x^1\mathbf{x}_1 + x^2\mathbf{x}_2 + \cdots + x^n\mathbf{x}_n)\cdot\mathbf{x}^i = x^i. \tag{6}$$

In fact, $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ and $\{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^n\}$ are (mutually) reciprocal.

To construct a reciprocal basis $(\mathbf{x}^1, \mathbf{x}^2)$ from a given basis $(\mathbf{x}_1, \mathbf{x}_2)$, we observe that we can always write

$$\begin{aligned}
\mathbf{x}_1 &= x_{11}\mathbf{x}^1 + x_{12}\mathbf{x}^2 \\
\mathbf{x}_2 &= x_{21}\mathbf{x}^1 + x_{22}\mathbf{x}^2 \\
\mathbf{x}^1 &= x^{11}\mathbf{x}_1 + x^{12}\mathbf{x}_2 \\
\mathbf{x}^2 &= x^{21}\mathbf{x}_1 + x^{22}\mathbf{x}_2
\end{aligned}$$

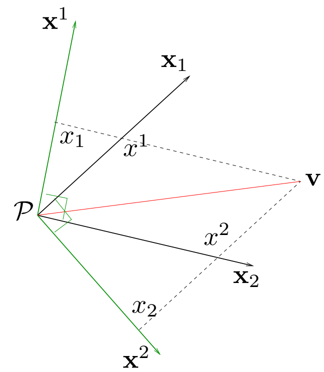

Figure 2: Basis vectors (in black) and reciprocal basis vectors (in green).

From Eq. 4 we have

$$\begin{aligned}
\mathbf{x}^1 \cdot \mathbf{x}_1 &= x_{11}x^{11} + x_{12}x^{12} = 1 \\
\mathbf{x}^1 \cdot \mathbf{x}_2 &= x_{11}x^{21} + x_{12}x^{22} = 0 \\
\mathbf{x}^2 \cdot \mathbf{x}_1 &= x_{21}x^{11} + x_{22}x^{12} = 0 \\
\mathbf{x}^2 \cdot \mathbf{x}_2 &= x_{21}x^{21} + x_{22}x^{22} = 1.
\end{aligned}$$

Knowing x_{11}, x_{12}, x_{21}, and x_{22}, we solve the system of linear equations and get

$$\begin{aligned}
x^{11} &= \frac{x_{22}}{x_{11}x_{22} - x_{12}x_{21}} \\
x^{12} &= \frac{-x_{21}}{x_{11}x_{22} - x_{12}x_{21}} \\
x^{21} &= \frac{-x_{21}}{x_{11}x_{22} - x_{12}x_{21}} \\
x^{12} &= \frac{x_{11}}{x_{11}x_{22} - x_{12}x_{21}}
\end{aligned} \tag{7}$$

Finally, we remark that the formulation of contravariance and covariance is often more natural in applications, in which there is a coordinate system, whose axes are laid on the tangent plane of a continuously deforming object and to which the material element of the object is fixed, as we will see in Section 4. This reference system is called *convected curvilinear system*, i.e. it is convected with the moving medium for describing its mechanics, and the coordinates of a point are called *curvilinear coordinates* or *convected coordinates*.

2.2 Permutation Symbol

For later reference, we introduce the *permutation symbol* e_{ijk}. It can be defined as the scalar triple product of unit vectors in a right-handed coordinate system $e_{ijk} = \mathbf{x}_i \cdot (\mathbf{x}_j \times \mathbf{x}_k)$

$$e_{ijk} = \left\| \begin{array}{ccc} x_{i1} & x_{i2} & x_{i3} \\ x_{j1} & x_{j2} & x_{j3} \\ x_{k1} & x_{k2} & x_{k3} \end{array} \right\| = \begin{cases} +1, \text{ if } (i,j,k) = (1,2,3), (3,1,2) \text{ or } (2,3,1) \\ -1, \text{ if } (i,j,k) = (1,3,2), (3,2,1) \text{ or } (2,1,3) \\ 0, \text{ otherwise} \end{cases}$$

This concept is extensible to arbitraty non-unitary basis $(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k)$, also referred as an ϵ-*system*

$$\epsilon_{ijk} = \left\| \begin{array}{ccc} a_{i1} & a_{i2} & a_{i3} \\ a_{j1} & a_{j2} & a_{j3} \\ a_{k1} & a_{k2} & a_{k3} \end{array} \right\| = \begin{cases} +\sqrt{a}, \text{ if } (i,j,k) = (1,2,3), (3,1,2) \text{ or } (2,3,1) \\ -\sqrt{a}, \text{ if } (i,j,k) = (1,3,2), (3,2,1) \text{ or } (2,1,3) \\ 0, \text{ otherwise} \end{cases},$$

where a is the determinant of the square matrix.

2.3 Tensors

We may think of a tensor as a generalization of vectors: a set of quantities that describe a geometrical object or a physical phenomenon and these quantities transform in a specific way under a change of coordinate system with respect to which they are defined. For example, a circle may be represented in rectangular coordinates (x, y) as

$$x^2 + y^2 = R^2$$

or in polar coordinates u as

$$x = R \cos u \qquad y = R \sin u.$$

A drastic change in the description of the circle is nothing but a change of coordinates. The quantities (x, y) and u, may be transformed on each other in accordance with a linear mapping. As explained in Section 2.1, the quantitative description of geometric or physical entities may change in the same or in the opposite direction as the change of basis vectors. If they change in the same direction, the tensor is said to be *contravariant*; otherwise it is *covariant*. A *notation of superscript and subscript* is adopted for distinguishing the components of a contravariant tensor from the components of a covariant one. Upper indices are used to indicate contravariant components, while lower indices are used to indicate covariant components. There exist, however, quantitative description that is independent of the coordinate system. It is called *invariant*. Roughly speaking, the product of a contravariant vector and a covariant vector always is an invariant.

Often the quantities used to express geometrical objects or physical phenomena are organized in arrays. The *order* (or degree) of a tensor is the dimensionality of the array needed to represent them. A scalar is representable by a number, which is a 0-dimensional array, so it is a *0^{th}-order tensor*. A vector follows the basis transformation rule as shown in Section and is representable by a 1-dimensional array, then it is a *1^{st}-order tensor*. A 2-dimensional array, or square matrix, is needed to represent a *2^{nd}-order tensor*. In general, an order-k tensor can be represented as a k-dimensional array of components. To avoid a cumbersome representation a tensor is usually stated in terms of its components. For example, instead of speaking of a n-dimensional vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ whose components are v_i, we may simply refer to the vector v_i. In Section 2.4 some examples of 2^{nd}-order tensor are given.

2.4 Metric and Curvature Tensors

In this section we present a summary of some concepts of our interest. It is based on the classical textbooks of Differential Geometry [43, 9, 4, 29].

We may represent a surface $\mathcal{S}$ as a net of parametric curves, such that every point $\mathcal{P}$ on the surface is a crossing point of two of them. Mathematically, we may express $\mathcal{S}$ as a function $\mathbf{r}(u, v) = (x(u, v),$ $y(u, v)$, $z(u, v))$ that maps a point (u, v) in a certain closed interval Ω onto a three-dimensional space $\Re^3$. The real variables u, v are called *coordinates* on $\mathcal{S}$. When we keep v constant, $\mathbf{r}(u, v_{constant})$ depends on only one parameter u and thus determine a curve on $\mathcal{S}$. Similarly, when we keep u constant, $\mathbf{r}(u_{constant}, v)$ represents another parametric curve. At $\mathcal{P}$ the vector $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$ is tangent to the curve $\mathbf{r}(u, v_{constant})$, and $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$ is tangent to the curve $\mathbf{r}(u_{constant}, v)$. If the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ do not vanish and have different directions, that is $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v \neq 0$, at every point, we say that $\mathbf{r}(u, v)$ is a *regular surface* (Figure 3).

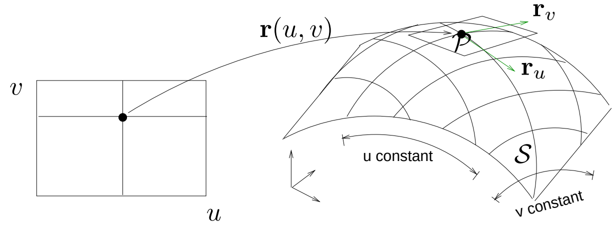

Figure 3: Parametrization of a surface.

2.4.1 Metric Tensors

Let $\alpha(t) = \mathbf{r}(u(t), v(t))$ be a curve on $\mathcal{S}$ that passes through $\mathcal{P}$. The vector at $\mathcal{P}$

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}$$

is tangent to the curve $\alpha(t)$ and therefore to $\mathcal{S}$, which may be rewritten in a form independent of the choice of parameter

$$d\mathbf{r} = \mathbf{r}_u du + \mathbf{r}_v dv. \quad (8)$$

In fact, all directional vectors $d_{\vec{d}}\mathbf{r} = \mathbf{v}$ at $\mathcal{P}$ in a direction $\vec{d}$ are linear combinations of $\mathbf{r}_u$ and $\mathbf{r}_v$ and thus they lie in the plane spanned by these two vectors. We say that this plane is the *tangent plane* at $\mathcal{P}$ to $\mathcal{S}$, which has the highest contact with $\mathcal{S}$, and any vector $\mathbf{v}$ in it is defined with respect to the local basis $\{\mathbf{r}_u, \mathbf{r}_v\}$.

The infinitesimal squared length $I(\alpha(t))$ of an *element of arc* of $\alpha(t)$ in the vicinity of $\mathcal{P}$ can be expressed by

$$\begin{aligned} I(\alpha(t)) &= d\alpha(t) \cdot d\alpha(t) \equiv a_{11}\frac{du}{dt}\frac{du}{dt} + a_{12}\frac{du}{dt}\frac{dv}{dt} + a_{21}\frac{du}{dt}\frac{dv}{dt} + a_{22}\frac{dv}{dt}\frac{dv}{dt} \\ &\equiv \begin{bmatrix} \frac{du}{dt} & \frac{dv}{dt} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix}, \end{aligned} \quad (9)$$

where

$$a_{11} = \mathbf{r}_u \cdot \mathbf{r}_u \qquad a_{12} = a_{21} = \mathbf{r}_u \cdot \mathbf{r}_v \qquad a_{22} = \mathbf{r}_v \cdot \mathbf{r}_v. \quad (10)$$

The quadratic or the bilinear form, given in Eq. 9, is the *first fundamental form*. Its *discriminant* $(a_{11}a_{22} - a_{12}^2)$ is always positive. Observe that we have $a_{12} = a_{21}$, because the dot product must be symmetric at every point. Further, the parametric curves should not be orthogonal. Only if $a_{12} = a_{21} = 0$, they are orthogonal.

When we introduce new coordinates under an affine coordinate transformation

$$\tilde{u} = \tilde{u}(u, v) \qquad \tilde{v} = \tilde{v}(u, v), \quad (11)$$

we have

$$\begin{bmatrix} \mathbf{r}_u & \mathbf{r}_v \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{\tilde{u}} & \mathbf{r}_{\tilde{v}} \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{\tilde{u}} & \mathbf{r}_{\tilde{v}} \end{bmatrix} J(u, v) \quad (12)$$

The matrix of all first-order partial derivatives $J(u, v)$ is known as *Jacobian matrix* of the transformation from the coordinates (u, v) to $(\tilde{u}, \tilde{v})$ and it may be thought as describing the orientation of a tangent plane to the surface $\mathcal{S}$ at a given point.

Conversely,

$$\begin{bmatrix} \mathbf{r}_{\tilde{u}} & \mathbf{r}_{\tilde{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u & \mathbf{r}_v \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u & \mathbf{r}_v \end{bmatrix} J(\tilde{u}, \tilde{v}) \quad (13)$$

Replacing it in Eq. 12 we get the inverse of the Jacobian matrix

$$J^{-1}(u, v) = J(\tilde{u}, \tilde{v}) = \begin{bmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{bmatrix} \quad (14)$$

This result is expected, since according to the inverse function theorem, the matrix inverse of the Jacobian matrix of a function is the Jacobian matrix of the inverse function. In consequence, the inverse of the Jacobian determinant of a transformation is the Jacobian determinant of the inverse transformation.

Plugging Eq. 12 into Eq. 8 we have the quantities $\tilde{a}_{11}$, $\tilde{a}_{12}$, $\tilde{a}_{21}$, and $\tilde{a}_{22}$ of the first fundamental form with respect to $\tilde{u}$, $\tilde{v}$

$$\begin{aligned} \tilde{a}_{11} &= a_{11}\frac{\partial u}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{u}} + a_{12}\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{u}} + a_{21}\frac{\partial v}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{u}} + a_{22}\frac{\partial v}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{u}} \\ \tilde{a}_{12} &= a_{11}\frac{\partial u}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{v}} + a_{12}\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} + a_{21}\frac{\partial v}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{v}} + a_{22}\frac{\partial v}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} \\ \tilde{a}_{21} &= a_{11}\frac{\partial u}{\partial \tilde{v}}\frac{\partial u}{\partial \tilde{u}} + a_{12}\frac{\partial u}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{u}} + a_{21}\frac{\partial v}{\partial \tilde{v}}\frac{\partial u}{\partial \tilde{u}} + a_{22}\frac{\partial v}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{u}} \\ \tilde{a}_{22} &= a_{11}\frac{\partial u}{\partial \tilde{v}}\frac{\partial u}{\partial \tilde{v}} + a_{12}\frac{\partial u}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{v}} + a_{21}\frac{\partial v}{\partial \tilde{v}}\frac{\partial u}{\partial \tilde{v}} + a_{22}\frac{\partial v}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{v}}. \end{aligned} \quad (15)$$

Rearranging Eq. 15 to matrix form and comparing it with Eq. 13, we may see that the components of the first fundamental form vary in the same direction as the change of basis

$$\begin{aligned}\begin{bmatrix}\tilde{a}_{11} & \tilde{a}_{12}\\ \tilde{a}_{21} & \tilde{a}_{22}\end{bmatrix} &= \begin{bmatrix}\frac{\partial u}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{u}}\\ \frac{\partial u}{\partial \tilde{v}} & \frac{\partial v}{\partial \tilde{v}}\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}}\\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}}\end{bmatrix}\\ &= J(\tilde{u},\tilde{v})^T \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} J(\tilde{u},\tilde{v}). \end{aligned} \tag{16}$$

Moreover, the components of the first fundamentl form enable us to measure lengths, angles, and areas in a surface; by them the "metric" in a surface is completely determined. Hence, they are also called *metric coefficients*, and because they vary in the same direction as the change of basis they are organized in 2-dimensional matrix denominated *covariant metric tensor (of second order)*.

One metric question that may arise is how the metric quantities behave with respect to the metric coefficients under coordinate transformations. How is, for example, the variation of the element arc under a coordinate transformation? Plugging the contravariant transformation of du and dv in Eq. 9

$$\begin{bmatrix}du\\ dv\end{bmatrix} = \begin{bmatrix}\frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}}\\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}}\end{bmatrix}\begin{bmatrix}d\tilde{u}\\ d\tilde{v}\end{bmatrix} \tag{17}$$

we get

$$d\mathbf{r}(\alpha(t)) \cdot d\mathbf{r}(\alpha(t)) = \tilde{a}_{11}d\tilde{u}d\tilde{u} + \tilde{a}_{12}d\tilde{u}d\tilde{v} + \tilde{a}_{21}d\tilde{u}d\tilde{v} + \tilde{a}_{22}d\tilde{v}d\tilde{v},$$

which leads us to conclude that the first fundamental form is invariant under coordinate transformations. We remark that the contravariant vector has the same transformation behavior as the differentials of the coordinates.

Another important metric quantity is the area. Is it invariant with respect to affine coordinate transformations? The function $\|\mathbf{r}_u \times \mathbf{r}_u\|$, expressable in terms of the discriminant of the tensor metric

$$\begin{aligned}&\|\mathbf{r}_u \times \mathbf{r}_v\|^2 + (\mathbf{r}_u \cdot \mathbf{r}_v)^2 = \|\mathbf{r}_u\|^2\,\|\mathbf{r}_v\|^2\\ &\Longrightarrow \|\mathbf{r}_u \times \mathbf{r}_v\|^2 = (a_{11}a_{22} - a_{12}a_{21}) = a\\ &\Longrightarrow \|\mathbf{r}_u \times \mathbf{r}_v\| = (a_{11}a_{22} - a_{12}a_{21})^{\frac{1}{2}} = a^{\frac{1}{2}},\end{aligned} \tag{18}$$

provides the area of the parallelogram generated by the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. The area of the bounded region of $\mathcal{S}$ defined in Ω is given by the expression

$$\int\int_\Omega \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv = \int\int_\Omega (a_{11}a_{22} - a_{12}^2)^{\frac{1}{2}} dudv. \tag{19}$$

Under the change of basis, the expression becomes

$$\begin{aligned}\int\int_\Omega \|\mathbf{r}_{\tilde{u}} \times \mathbf{r}_{\tilde{v}}\|\, d\tilde{u}d\tilde{v} &= \int\int_\Omega (\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2)^{\frac{1}{2}} d\tilde{u}d\tilde{v}\\ = \int\int_\Omega ((\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} - \frac{\partial u}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{u}})(a_{11}a_{22} - a_{12}^2)^{\frac{1}{2}})&((\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial u})dudv)\\ &= \int\int_\Omega (a_{11}a_{22} - a_{12}^2)^{\frac{1}{2}} dudv.\end{aligned}$$

Observe that the area of $\mathcal{S}$ in Ω continues to hold. This is because that, when carrying out the transformation, the integrand of Eq. 19 is multiplied by the determinant of the Jacobian of the transformation, while $\|\mathbf{r}_u \times \mathbf{r}_u\|$ is multiplied by the absolute value of the determinant of the Jacobian corresponding to the inverse transformation.

The tangent plane at $\mathcal{P}$ on the surface $\mathcal{S}$ is equally well spanned by the *reciprocal basis vectors* $\mathbf{r}^u$ and $\mathbf{r}^v$ that satisfy the following properties:

$$\mathbf{r}^u \cdot \mathbf{r}_u = 1 \qquad \mathbf{r}^u \cdot \mathbf{r}_v = \mathbf{r}^v \cdot \mathbf{r}_v = 0 \qquad \mathbf{r}^v \cdot \mathbf{r}_v = 1 \tag{20}$$

Figure 4 depicts the relationship between the contravariant and covariant basis vectors. In praticular, we may write each contravariant basis vector as a linear combination of covariant basis vectors

$$\begin{aligned}\mathbf{r}^u &= a^{11}\mathbf{r}_u + a^{12}\mathbf{r}_v\\ \mathbf{r}^v &= a^{21}\mathbf{r}_u + a^{22}\mathbf{r}_v\end{aligned} \tag{21}$$

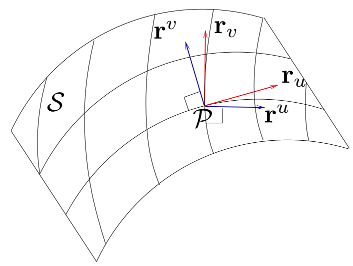

Figure 4: Contravariant and covariant vectors on the tangent plane.

and take the dot product of the expressions and $\mathbf{r}_u$ and $\mathbf{r}_v$ respectively, to get

$$\begin{aligned}
\mathbf{r}^u \cdot \mathbf{r}_u &= 1 = a^{11}\mathbf{r}_u \cdot \mathbf{r}_u + a^{12}\mathbf{r}_v \cdot \mathbf{r}_u = a^{11}a_{11} + a^{12}a_{12} \\
\mathbf{r}^u \cdot \mathbf{r}_v &= 0 = a^{11}\mathbf{r}_u \cdot \mathbf{r}_v + a^{12}\mathbf{r}_v \cdot \mathbf{r}_v = a^{11}a_{12} + a^{12}a_{22} \\
\mathbf{r}^v \cdot \mathbf{r}_u &= 0 = a^{21}\mathbf{r}_u \cdot \mathbf{r}_u + a^{22}\mathbf{r}_v \cdot \mathbf{r}_u = a^{21}a_{11} + a^{22}a_{12} \\
\mathbf{r}^v \cdot \mathbf{r}_v &= 1 = a^{21}\mathbf{r}_u \cdot \mathbf{r}_v + a^{22}\mathbf{r}_v \cdot \mathbf{r}_v = a^{21}a_{21} + a^{22}a_{22}.
\end{aligned} \tag{22}$$

Solving this linear system we have

$$a^{11} = \frac{a_{22}}{a} \qquad a^{12} = a^{21} = -\frac{a_{12}}{a} \qquad a^{22} = \frac{a_{11}}{a} \tag{23}$$

where a is defined in Eq. 18.

Now taking the dot product of Eq. 21 and $\mathbf{r}^u$ or $\mathbf{r}^v$, and applying Eq. 20, we obtain

$$\begin{aligned}
\mathbf{r}^u \cdot \mathbf{r}^u &= a^{11}\mathbf{r}_u \cdot \mathbf{r}^u + a^{12}\mathbf{r}_v \cdot \mathbf{r}^u = a^{11} \\
\mathbf{r}^v \cdot \mathbf{r}^v &= a^{21}\mathbf{r}_u \cdot \mathbf{r}^v + a^{22}\mathbf{r}_v \cdot \mathbf{r}^v = a^{22} \\
\mathbf{r}^u \cdot \mathbf{r}^v &= a^{11}\mathbf{r}_u \cdot \mathbf{r}^v + a^{12}\mathbf{r}_v \cdot \mathbf{r}^v = a^{12} \\
\mathbf{r}^v \cdot \mathbf{r}^u &= a^{21}\mathbf{r}_u \cdot \mathbf{r}^u + a^{22}\mathbf{r}_v \cdot \mathbf{r}^u = a^{21}
\end{aligned} \tag{24}$$

Solving Eq. 21, we also get the following relations

$$\begin{aligned}
\mathbf{r}_u &= a_{11}\mathbf{r}^u + a_{12}\mathbf{r}^v \\
\mathbf{r}_v &= a_{21}\mathbf{r}^u + a_{22}\mathbf{r}^v
\end{aligned} \tag{25}$$

If we change the coordinates to $(\tilde{u}, \tilde{v})$, we have after some algebric manipulations

$$\begin{aligned}
\tilde{a}^{11} &= a^{11}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{u}}{\partial u} + a^{12}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{u}}{\partial v} + a^{21}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{u}}{\partial u} + a^{22}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{u}}{\partial v} \\
\tilde{a}^{12} &= a^{11}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial u} + a^{12}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial v} + a^{21}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial u} + a^{22}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial v} \\
\tilde{a}^{21} &= a^{11}\frac{\partial \tilde{v}}{\partial u}\frac{\partial \tilde{u}}{\partial u} + a^{12}\frac{\partial \tilde{v}}{\partial u}\frac{\partial \tilde{u}}{\partial v} + a^{21}\frac{\partial \tilde{v}}{\partial v}\frac{\partial \tilde{u}}{\partial u} + a^{22}\frac{\partial \tilde{v}}{\partial v}\frac{\partial \tilde{u}}{\partial v} \\
\tilde{a}^{22} &= a^{11}\frac{\partial \tilde{v}}{\partial u}\frac{\partial \tilde{v}}{\partial u} + a^{12}\frac{\partial \tilde{v}}{\partial u}\frac{\partial \tilde{v}}{\partial v} + a^{21}\frac{\partial \tilde{v}}{\partial v}\frac{\partial \tilde{v}}{\partial u} + a^{22}\frac{\partial \tilde{v}}{\partial v}\frac{\partial \tilde{v}}{\partial v}.
\end{aligned} \tag{26}$$

and we may see that the quantities a^{11}, a^{12}, a^{21} and a^{22} vary in the opposite direction as the change of basis. We than say that they build a *contravariant metric tensor*. It is worth remarking that the covariant metric tensors and the contravariant metric tensors are *conjugate*, for

$$a^{\alpha 1}a_{1\beta} + a^{\alpha 2}a_{2\beta} = \delta^{\alpha}_{\beta} = \begin{cases} 0, \text{ if } \alpha \neq \beta \\ 1, \text{ if } \alpha = \beta \end{cases} \qquad \alpha, \beta = \{1, 2\}. \tag{27}$$

2.4.2 Christoffel Symbols

Let $(\mathbf{r}_u, \mathbf{r}_v, \mathbf{n})$ be a (local) reference basis at a particular $\mathcal{P}$. If we move in a particular direction $\mathbf{d}$, this reference changes according to the derivatives $d\mathbf{r}_u$ and $d\mathbf{r}_v$ along $\mathbf{d}$. Any changes in a vector with respect to this local reference basis should, therefore, be "adjusted" with the basis changes for ensuring that the results are consistent in the ambient space. The Christoffel symbols are three-index quantities that serve to convey these changes.

Expressing the second-order partial derivatives of $\mathbf{r}$ as a linear combination of the reference $\{\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}\}$ (with respect to a locally flat coordinates at a particular point $\mathcal{P}$)

$$\begin{aligned}
\frac{\partial^2 \mathbf{r}}{\partial u \partial u} = \mathbf{r}_{uu} &= \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + b_{11}\mathbf{n} \\
\frac{\partial^2 \mathbf{r}}{\partial u \partial v} = \mathbf{r}_{uv} &= \Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + b_{12}\mathbf{n} \\
\frac{\partial^2 \mathbf{r}}{\partial v \partial u} = \mathbf{r}_{vu} &= \Gamma^1_{21}\mathbf{r}_u + \Gamma^2_{21}\mathbf{r}_v + b_{21}\mathbf{n} \\
\frac{\partial^2 \mathbf{r}}{\partial v \partial v} = \mathbf{r}_{vv} &= \Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + b_{22}\mathbf{n},
\end{aligned} \tag{28}$$

we may evaluate how the basis vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ change along a curve.

One way to determine the coefficients $\Gamma^\lambda_{\alpha\beta}$ is to first take the dot product of the first-order derivative of $\mathbf{r}$ and both sides of Eq. 28

$$\begin{aligned}
\mathbf{r}_{uu}\cdot\mathbf{r}_u &= \Gamma^1_{11}\mathbf{r}_u\cdot\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v\cdot\mathbf{r}_u + b_{11}\mathbf{n}\cdot\mathbf{r}_u = \Gamma^1_{11}a_{11} + \Gamma^2_{11}a_{21} \\
\mathbf{r}_{uu}\cdot\mathbf{r}_v &= \Gamma^1_{11}\mathbf{r}_u\cdot\mathbf{r}_v + \Gamma^2_{11}\mathbf{r}_v\cdot\mathbf{r}_v + b_{11}\mathbf{n}\cdot\mathbf{r}_v = \Gamma^1_{11}a_{12} + \Gamma^2_{11}a_{22} \\
\mathbf{r}_{uv}\cdot\mathbf{r}_u &= \Gamma^1_{12}\mathbf{r}_u\cdot\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v\cdot\mathbf{r}_u + b_{12}\mathbf{n}\cdot\mathbf{r}_u = \Gamma^1_{12}a_{11} + \Gamma^2_{12}a_{21} \\
\mathbf{r}_{uv}\cdot\mathbf{r}_v &= \Gamma^1_{12}\mathbf{r}_u\cdot\mathbf{r}_v + \Gamma^2_{12}\mathbf{r}_v\cdot\mathbf{r}_v + b_{12}\mathbf{n}\cdot\mathbf{r}_v = \Gamma^1_{12}a_{12} + \Gamma^2_{12}a_{22} \\
\mathbf{r}_{vu}\cdot\mathbf{r}_u &= \Gamma^1_{21}\mathbf{r}_u\cdot\mathbf{r}_u + \Gamma^2_{21}\mathbf{r}_v\cdot\mathbf{r}_u + b_{21}\mathbf{n}\cdot\mathbf{r}_u = \Gamma^1_{21}a_{11} + \Gamma^2_{21}a_{21} \\
\mathbf{r}_{vu}\cdot\mathbf{r}_v &= \Gamma^1_{21}\mathbf{r}_u\cdot\mathbf{r}_v + \Gamma^2_{21}\mathbf{r}_v\cdot\mathbf{r}_v + b_{21}\mathbf{n}\cdot\mathbf{r}_v = \Gamma^1_{21}a_{12} + \Gamma^2_{21}a_{22} \\
\mathbf{r}_{vv}\cdot\mathbf{r}_u &= \Gamma^1_{22}\mathbf{r}_u\cdot\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v\cdot\mathbf{r}_u + b_{22}\mathbf{n}\cdot\mathbf{r}_u = \Gamma^1_{22}a_{11} + \Gamma^2_{22}a_{21} \\
\mathbf{r}_{vv}\cdot\mathbf{r}_v &= \Gamma^1_{22}\mathbf{r}_u\cdot\mathbf{r}_v + \Gamma^2_{22}\mathbf{r}_v\cdot\mathbf{r}_v + b_{22}\mathbf{n}\cdot\mathbf{r}_v = \Gamma^1_{22}a_{12} + \Gamma^2_{22}a_{22}
\end{aligned} \tag{29}$$

and then multiply the both sides by the coefficients of the contravariant metric tensor (Eq. 24) and summing them. Using further the fact that the covariant and contravariant metric tensors are conjugate, we obtain (Eq. 27)

$$\begin{aligned}
&\mathbf{r}_{uu}\cdot\mathbf{r}_u a^{11} + \mathbf{r}_{uu}\cdot\mathbf{r}_v a^{12} = \mathbf{r}_{uu}\cdot(\mathbf{r}_u a^{11} + \mathbf{r}_v a^{12}) = \mathbf{r}_{uu}\cdot\mathbf{r}^u \\
&= \Gamma^1_{11}a_{11}a^{11} + \Gamma^2_{11}a_{21}a^{11} + \Gamma^1_{11}a_{21}a^{12} + \Gamma^2_{11}a_{22}a^{12} = \Gamma^1_{11}(a_{11}a^{11} + a_{21}a^{12}) = \Gamma^1_{11} \\
&\mathbf{r}_{uu}\cdot\mathbf{r}_u a^{21} + \mathbf{r}_{uu}\cdot\mathbf{r}_v a^{22} = \mathbf{r}_{uu}\cdot(\mathbf{r}_u a^{21} + \mathbf{r}_v a^{22}) = \mathbf{r}_{uu}\cdot\mathbf{r}^v \\
&= \Gamma^2_{11}a_{11}a^{21} + \Gamma^2_{11}a_{12}a^{21} + \Gamma^2_{11}a_{21}a^{22} + \Gamma^2_{11}a_{22}a^{22} = \Gamma^2_{11}(a_{12}a^{21} + a_{22}a^{22}) = \Gamma^2_{11} \\
&\mathbf{r}_{uv}\cdot\mathbf{r}_u a^{11} + \mathbf{r}_{uv}\cdot\mathbf{r}_v a^{12} = \mathbf{r}_{uv}\cdot(\mathbf{r}_u a^{11} + \mathbf{r}_v a^{12}) = \mathbf{r}_{uv}\cdot\mathbf{r}^u \\
&= \Gamma^1_{12}a_{11}a^{11} + \Gamma^2_{12}a_{21}a^{11} + \Gamma^1_{12}a_{21}a^{12} + \Gamma^2_{12}a_{22}a^{12} = \Gamma^1_{12}(a_{11}a^{11} + a_{21}a^{12}) = \Gamma^1_{12} \\
&\mathbf{r}_{uv}\cdot\mathbf{r}_u a^{21} + \mathbf{r}_{uv}\cdot\mathbf{r}_v a^{22} = \mathbf{r}_{uv}\cdot(\mathbf{r}_u a^{21} + \mathbf{r}_v a^{22}) = \mathbf{r}_{uv}\cdot\mathbf{r}^v \\
&= \Gamma^2_{12}a_{11}a^{21} + \Gamma^2_{12}a_{12}a^{21} + \Gamma^2_{12}a_{21}a^{22} + \Gamma^2_{12}a_{22}a^{22} = \Gamma^2_{12}(a_{12}a^{21} + a_{22}a^{22}) = \Gamma^2_{12} \\
&\mathbf{r}_{vu}\cdot\mathbf{r}_u a^{11} + \mathbf{r}_{vu}\cdot\mathbf{r}_v a^{12} = \mathbf{r}_{vu}\cdot(\mathbf{r}_u a^{11} + \mathbf{r}_v a^{12}) = \mathbf{r}_{vu}\cdot\mathbf{r}^u \\
&= \Gamma^1_{21}a_{11}a^{11} + \Gamma^2_{21}a_{21}a^{11} + \Gamma^1_{21}a_{21}a^{12} + \Gamma^2_{21}a_{22}a^{12} = \Gamma^1_{21}(a_{11}a^{11} + a_{21}a^{12}) = \Gamma^1_{21} \\
&\mathbf{r}_{vu}\cdot\mathbf{r}_u a^{21} + \mathbf{r}_{uv}\cdot\mathbf{r}_v a^{22} = \mathbf{r}_{vu}\cdot(\mathbf{r}_u a^{21} + \mathbf{r}_v a^{22}) = \mathbf{r}_{vu}\mathbf{r}^v \\
&= \Gamma^2_{21}a_{11}a^{21} + \Gamma^2_{21}a_{12}a^{21} + \Gamma^2_{21}a_{21}a^{22} + \Gamma^2_{21}a_{22}a^{22} = \Gamma^2_{21}(a_{12}a^{21} + a_{22}a^{22}) = \Gamma^2_{21} \\
&\mathbf{r}_{vv}\cdot\mathbf{r}_u a^{11} + \mathbf{r}_{vv}\cdot\mathbf{r}_v a^{12} = \mathbf{r}_{vv}\cdot(\mathbf{r}_u a^{11} + \mathbf{r}_v a^{12}) = \mathbf{r}_{vv}\cdot\mathbf{r}^u \\
&= \Gamma^1_{22}a_{11}a^{11} + \Gamma^2_{22}a_{21}a^{11} + \Gamma^1_{22}a_{21}a^{12} + \Gamma^2_{22}a_{22}a^{12} = \Gamma^1_{22}(a_{11}a^{11} + a_{21}a^{12}) = \Gamma^1_{22} \\
&\mathbf{r}_{vv}\cdot\mathbf{r}_u a^{21} + \mathbf{r}_{vv}\cdot\mathbf{r}_v a^{22} = \mathbf{r}_{vv}\cdot(\mathbf{r}_u a^{21} + \mathbf{r}_v a^{22}) = \mathbf{r}_{vv}\cdot\mathbf{r}^v \\
&= \Gamma^2_{22}a_{11}a^{21} + \Gamma^2_{22}a_{12}a^{21} + \Gamma^2_{22}a_{21}a^{22} + \Gamma^2_{22}a_{22}a^{22} = \Gamma^2_{22}(a_{12}a^{21} + a_{22}a^{22}) = \Gamma^2_{22}.
\end{aligned} \tag{30}$$

The coefficients $\Gamma^\lambda_{\alpha\beta}$ are, in fact, the scalar products of the second derivatives onto the contravariant basis vectors. They are called *Christoffel symbols (of the second kind)* and provide connection between the local tangent planes, making thus possible the computation of the differences between two vectors in two nearby points on a surface of arbitrary geometry as explained in Section 2.4.4.

If we call the scalar products of the first and the second derivatives of the covariant basis vectors in Eq. 29 the *Christoffel symbols of the first kind* and denote them by $\Gamma_{\alpha\beta\lambda}$, namely

$$\begin{aligned}
\Gamma_{111} &= \mathbf{r}_{uu} \cdot \mathbf{r}_u \\
\Gamma_{112} &= \mathbf{r}_{uu} \cdot \mathbf{r}_v \\
\Gamma_{121} &= \mathbf{r}_{uv} \cdot \mathbf{r}_u \\
\Gamma_{122} &= \mathbf{r}_{uv} \cdot \mathbf{r}_v \\
\Gamma_{211} &= \mathbf{r}_{vu} \cdot \mathbf{r}_u \\
\Gamma_{212} &= \mathbf{r}_{vu} \cdot \mathbf{r}_v \\
\Gamma_{221} &= \mathbf{r}_{vv} \cdot \mathbf{r}_u \\
\Gamma_{222} &= \mathbf{r}_{vv} \cdot \mathbf{r}_v,
\end{aligned} \tag{31}$$

we have the following relations between the Christoffel symbols of the first kind and of the second kind

$$\Gamma_{\alpha\beta\gamma} = a_{1\gamma}\Gamma^1_{\alpha\beta} + a_{2\gamma}\Gamma^2_{\alpha\beta}, \quad with \quad \alpha, \beta \in \{1, 2\}. \tag{32}$$

Conversely, we may derive the Christoffel symbols of the second kind from the Christoffel symbols of the first kind and the components of the contravariant metric tensor

$$\Gamma^{\gamma}_{\alpha\beta} = a^{1\gamma}\Gamma_{\alpha\beta 1} + a^{2\gamma}\Gamma_{\alpha\beta 2}, \quad with \quad \alpha, \beta \in \{1, 2\}. \tag{33}$$

Observe that, bacause $\mathbf{r}_{vu} = \mathbf{r}_{uv}$, the Christofell symbols of the first kind and the second kind are symmetric with respect to the first two indices, i.e.,

$$\Gamma_{ijk} = \Gamma_{jik} \qquad \Gamma^k_{ij} = \Gamma^k_{ji}.$$

If we differentiate partially the metric tensors with respect to u and v, we have

$$\begin{aligned}
\frac{\partial a_{11}}{\partial u} &= \mathbf{r}_{uu} \cdot \mathbf{r}_u + \mathbf{r}_u \cdot \mathbf{r}_{uu} = 2\Gamma_{111} \\
\frac{\partial a_{12}}{\partial u} &= \mathbf{r}_{uu} \cdot \mathbf{r}_v + \mathbf{r}_u \cdot \mathbf{r}_{vu} = \Gamma_{112} + \Gamma_{211} \\
\frac{\partial a_{11}}{\partial v} &= \mathbf{r}_{uv} \cdot \mathbf{r}_u + \mathbf{r}_u \cdot \mathbf{r}_{uv} = 2\Gamma_{121} \\
\frac{\partial a_{12}}{\partial v} &= \mathbf{r}_{uv} \cdot \mathbf{r}_v + \mathbf{r}_u \cdot \mathbf{r}_{vv} = \Gamma_{122} + \Gamma_{221} \\
\frac{\partial a_{22}}{\partial u} &= \mathbf{r}_{vu} \cdot \mathbf{r}_v + \mathbf{r}_v \cdot \mathbf{r}_{vu} = 2\Gamma_{212} \\
\frac{\partial a_{22}}{\partial v} &= \mathbf{r}_{vv} \cdot \mathbf{r}_v + \mathbf{r}_v \cdot \mathbf{r}_{vv} = 2\Gamma_{222}.
\end{aligned} \tag{34}$$

In other words, the first partial derivatives of the components $a_{\alpha\beta}$ of the first fundamental form can be represented as a sum of two Christoffel symbols of the first kind (Eq. 35)

$$\frac{\partial a_{\alpha\beta}}{\partial u^{\lambda}} = \Gamma_{\alpha\lambda\beta} + \Gamma_{\beta\lambda\alpha} \tag{35}$$

Adding and subtracting appropriately these relations, we have

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial a_{jk}}{\partial coord(i)} + \frac{\partial a_{ki}}{\partial coord(j)} - \frac{\partial a_{ij}}{\partial coord(k)}\right), \tag{36}$$

where $coord(1) = u$ and $coord(2) = v$. It shows that the Christoffel symbols vanish in a coordinate system if and only if the metric tensor has constant components in that system, such as in the orthonormal coordinate system.

We may also express the partial derivatives of the contravariant basis vectors as linear combinations of themselves $\{\mathbf{r}^u, \mathbf{r}^v, \mathbf{n}\}$

$$\begin{aligned}
\frac{\partial \mathbf{r}^u}{\partial u} = \mathbf{r}^u_u &= \Upsilon^1_{11}\mathbf{r}^u + \Upsilon^1_{21}\mathbf{r}^v + h^1_1\mathbf{n} \\
\frac{\partial \mathbf{r}^u}{\partial v} = \mathbf{r}^u_v &= \Upsilon^1_{12}\mathbf{r}^u + \Upsilon^1_{22}\mathbf{r}^v + h^1_2\mathbf{n} \\
\frac{\partial \mathbf{r}^v}{\partial u} = \mathbf{r}^v_u &= \Upsilon^2_{11}\mathbf{r}^u + \Upsilon^2_{21}\mathbf{r}^v + h^2_1\mathbf{n} \\
\frac{\partial \mathbf{r}^v}{\partial v} = \mathbf{r}^v_v &= \Upsilon^2_{12}\mathbf{r}^v + \Upsilon^2_{22}\mathbf{r}^v + h^2_2\mathbf{n}
\end{aligned} \tag{37}$$

Taking the scalar product of $\mathbf{n}$ and both sides of Eq. 37 and knowing that $\mathbf{r}^u \cdot \mathbf{n} = \mathbf{r}^v \cdot \mathbf{n} = 0$ and $\mathbf{n} \cdot \mathbf{n} = 1$, we obtain

$$\begin{aligned}
\frac{\partial \mathbf{r}^u}{\partial u}\mathbf{n} &= \frac{\partial (a^{11}\mathbf{r}_u + a^{12}\mathbf{r}_v)}{\partial u}\mathbf{n} = (a^{11}\mathbf{r}_{uu} + a^{12}\mathbf{r}_{vu})\mathbf{n} = a^{11}\mathbf{r}_{uu}\mathbf{n} + a^{12}\mathbf{r}_{vu}\mathbf{n} \\
&= a^{11}b_{11} + a^{12}b_{21} = b_1^1 = h_1^1 \\
\frac{\partial \mathbf{r}^u}{\partial v}\mathbf{n} &= \frac{\partial (a^{11}\mathbf{r}_u + a^{12}\mathbf{r}_v)}{\partial v}\mathbf{n} = (a^{11}\mathbf{r}_{uv} + a^{12}\mathbf{r}_{vv})\mathbf{n} = a^{11}\mathbf{r}_{uv}\mathbf{n} + a^{12}\mathbf{r}_{vv}\mathbf{n} \\
&= a^{11}b_{12} + a^{12}b_{21} = b_2^1 = h_2^1 \\
\frac{\partial \mathbf{r}^v}{\partial u}\mathbf{n} &= \frac{\partial (a^{21}\mathbf{r}_u + a^{22}\mathbf{r}_v)}{\partial u}\mathbf{n} = (a^{21}\mathbf{r}_{uu} + a^{22}\mathbf{r}_{vu})\mathbf{n} = a^{21}\mathbf{r}_{uu}\mathbf{n} + a^{22}\mathbf{r}_{vu}\mathbf{n} \\
&= a^{21}b_{11} + a^{22}b_{21} = b_1^2 = h_1^2 \\
\frac{\partial \mathbf{r}^v}{\partial v}\mathbf{n} &= \frac{\partial (a^{21}\mathbf{r}_u + a^{22}\mathbf{r}_v)}{\partial v}\mathbf{n} = (a^{21}\mathbf{r}_{uv} + a^{22}\mathbf{r}_{vv})\mathbf{n} = a^{21}\mathbf{r}_{uv}\mathbf{n} + a^{22}\mathbf{r}_{vv}\mathbf{n} \\
&= a^{21}b_{12} + a^{22}b_{22} = b_2^2 = h_2^2.
\end{aligned}$$

Based on Eq. 22 we derive the equalities

$$\begin{aligned}
\frac{\partial (\mathbf{r}^u \cdot \mathbf{r}_u)}{\partial u} = 0 \quad &\Longrightarrow \quad \mathbf{r}_u^u \cdot \mathbf{r}_u + \mathbf{r}^u \cdot \mathbf{r}_{uu} = 0 \Longrightarrow \Gamma_{11}^1 = \mathbf{r}^u \cdot \mathbf{r}_{uu} = -\mathbf{r}_u^u \cdot \mathbf{r}_u \\
\frac{\partial (\mathbf{r}^u \cdot \mathbf{r}_u)}{\partial v} = 0 \quad &\Longrightarrow \quad \mathbf{r}_v^u \cdot \mathbf{r}_u + \mathbf{r}^u \cdot \mathbf{r}_{uv} = 0 \Longrightarrow \Gamma_{12}^1 = \mathbf{r}^u \cdot \mathbf{r}_{uv} = -\mathbf{r}_v^u \cdot \mathbf{r}_u \\
\frac{\partial (\mathbf{r}^u \cdot \mathbf{r}_v)}{\partial u} = 0 \quad &\Longrightarrow \quad \mathbf{r}_u^u \cdot \mathbf{r}_v + \mathbf{r}^u \cdot \mathbf{r}_{vu} = 0 \Longrightarrow \Gamma_{21}^1 = \mathbf{r}^u \cdot \mathbf{r}_{vu} = -\mathbf{r}_u^u \cdot \mathbf{r}_v \\
\frac{\partial (\mathbf{r}^u \cdot \mathbf{r}_v)}{\partial v} = 0 \quad &\Longrightarrow \quad \mathbf{r}_v^u \cdot \mathbf{r}_v + \mathbf{r}^u \cdot \mathbf{r}_{vv} = 0 \Longrightarrow \Gamma_{22}^1 = \mathbf{r}^u \cdot \mathbf{r}_{vv} = -\mathbf{r}_v^u \cdot \mathbf{r}_v \\
\frac{\partial (\mathbf{r}^v \cdot \mathbf{r}_u)}{\partial u} = 0 \quad &\Longrightarrow \quad \mathbf{r}_u^u \cdot \mathbf{r}_u + \mathbf{r}^u \cdot \mathbf{r}_{uu} = 0 \Longrightarrow \Gamma_{11}^2 = \mathbf{r}^v \cdot \mathbf{r}_{uu} = -\mathbf{r}_u^v \cdot \mathbf{r}_u \\
\frac{\partial (\mathbf{r}^v \cdot \mathbf{r}_u)}{\partial v} = 0 \quad &\Longrightarrow \quad \mathbf{r}_v^v \cdot \mathbf{r}_u + \mathbf{r}^v \cdot \mathbf{r}_{uv} = 0 \Longrightarrow \Gamma_{12}^2 = \mathbf{r}^v \cdot \mathbf{r}_{uv} = -\mathbf{r}_v^v \cdot \mathbf{r}_u \\
\frac{\partial (\mathbf{r}^v \cdot \mathbf{r}_v)}{\partial u} = 0 \quad &\Longrightarrow \quad \mathbf{r}_u^v \cdot \mathbf{r}_v + \mathbf{r}^v \cdot \mathbf{r}_{vu} = 0 \Longrightarrow \Gamma_{21}^2 = \mathbf{r}^v \cdot \mathbf{r}_{vu} = -\mathbf{r}_u^v \cdot \mathbf{r}_v \\
\frac{\partial (\mathbf{r}^v \cdot \mathbf{r}_v)}{\partial v} = 0 \quad &\Longrightarrow \quad \mathbf{r}_v^v \cdot \mathbf{r}_v + \mathbf{r}^v \cdot \mathbf{r}_{vv} = 0 \Longrightarrow \Gamma_{22}^2 = \mathbf{r}^v \cdot \mathbf{r}_{vv} = -\mathbf{r}_v^v \cdot \mathbf{r}_v,
\end{aligned} \tag{38}$$

from which the coefficients $\Upsilon_{\alpha\beta}^{\lambda}$ may be determined by taking the scalar product of the basis vectors and Eq. 37

$$\begin{aligned}
\mathbf{r}_u^u \cdot \mathbf{r}_u = -\Gamma_{11}^1 &= \Upsilon_{11}^1 \mathbf{r}^u \cdot \mathbf{r}_u + \Upsilon_{21}^1 \mathbf{r}^v \cdot \mathbf{r}_u + h_1^1 \mathbf{n} \cdot \mathbf{r}_u = \Upsilon_{11}^1 \\
\mathbf{r}_u^u \cdot \mathbf{r}_v = -\Gamma_{21}^1 &= \Upsilon_{11}^1 \mathbf{r}^u \cdot \mathbf{r}_v + \Upsilon_{21}^1 \mathbf{r}^v \cdot \mathbf{r}_v + h_1^1 \mathbf{n} \cdot \mathbf{r}_v = \Upsilon_{21}^1 \\
\mathbf{r}_v^u \cdot \mathbf{r}_u = -\Gamma_{12}^1 &= \Upsilon_{12}^1 \mathbf{r}^u \cdot \mathbf{r}_u + \Upsilon_{22}^1 \mathbf{r}^v \cdot \mathbf{r}_u + h_2^1 \mathbf{n} \cdot \mathbf{r}_u = \Upsilon_{12}^1 \\
\mathbf{r}_v^u \cdot \mathbf{r}_v = -\Gamma_{22}^1 &= \Upsilon_{12}^1 \mathbf{r}^u \cdot \mathbf{r}_v + \Upsilon_{22}^1 \mathbf{r}^v \cdot \mathbf{r}_v + h_2^1 \mathbf{n} \cdot \mathbf{r}_v = \Upsilon_{22}^1 \\
\mathbf{r}_u^v \cdot \mathbf{r}_u = -\Gamma_{11}^2 &= \Upsilon_{11}^2 \mathbf{r}^u \cdot \mathbf{r}_u + \Upsilon_{21}^2 \mathbf{r}^v \cdot \mathbf{r}_u + h_1^2 \mathbf{n} \cdot \mathbf{r}_u = \Upsilon_{11}^2 \\
\mathbf{r}_u^v \cdot \mathbf{r}_v = -\Gamma_{21}^2 &= \Upsilon_{11}^2 \mathbf{r}^u \cdot \mathbf{r}_v + \Upsilon_{21}^2 \mathbf{r}^v \cdot \mathbf{r}_v + h_1^2 \mathbf{n} \cdot \mathbf{r}_u = \Upsilon_{21}^2 \\
\mathbf{r}_v^v \cdot \mathbf{r}_u = -\Gamma_{12}^2 &= \Upsilon_{12}^2 \mathbf{r}^u \cdot \mathbf{r}_u + \Upsilon_{22}^2 \mathbf{r}^v \cdot \mathbf{r}_u + h_2^2 \mathbf{n} \cdot \mathbf{r}_u = \Upsilon_{12}^2 \\
\mathbf{r}_v^v \cdot \mathbf{r}_v = -\Gamma_{22}^2 &= \Upsilon_{12}^2 \mathbf{r}^u \cdot \mathbf{r}_v + \Upsilon_{22}^2 \mathbf{r}^v \cdot \mathbf{r}_v + h_2^2 \mathbf{n} \cdot \mathbf{r}_v = \Upsilon_{v2}^2.
\end{aligned}$$

They lead us to the relations

$$\begin{aligned}
\frac{\partial \mathbf{r}^u}{\partial u} = \mathbf{r}_u^u &= -\Gamma_{11}^1 \mathbf{r}^u - \Gamma_{21}^1 \mathbf{r}^v + b_1^1 \mathbf{n} \\
\frac{\partial \mathbf{r}^u}{\partial v} = \mathbf{r}_v^u &= -\Gamma_{12}^1 \mathbf{r}^u - \Gamma_{22}^1 \mathbf{r}^v + b_2^1 \mathbf{n} \\
\frac{\partial \mathbf{r}^v}{\partial u} = \mathbf{r}_u^v &= -\Gamma_{11}^2 \mathbf{r}^u - \Gamma_{21}^2 \mathbf{r}^v + b_1^2 \mathbf{n} \\
\frac{\partial \mathbf{r}^v}{\partial v} = \mathbf{r}_v^v &= -\Gamma_{12}^2 \mathbf{r}^v - \Gamma_{22}^2 \mathbf{r}^v + b_2^2 \mathbf{n}
\end{aligned} \tag{39}$$

We remark that if we change the coordinates to $(\tilde{u},\tilde{v})$ these symbols do not follow the transformation law of a tensor. According to Eq. 28, we have the following equalities with respect to the coordinates (u,v) and $(\tilde{u},\tilde{v})$, respectively:

$$\begin{aligned}
\mathbf{r}_{uu} &= \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + b_{11}\mathbf{n} \\
\mathbf{r}_{uv} &= \Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + b_{12}\mathbf{n} \\
\mathbf{r}_{vu} &= \Gamma^1_{21}\mathbf{r}_u + \Gamma^2_{21}\mathbf{r}_v + b_{21}\mathbf{n} \\
\mathbf{r}_{vv} &= \Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + b_{22}\mathbf{n} \\
\mathbf{r}_{\tilde{u}\tilde{u}} &= \tilde{\Gamma}^1_{11}\mathbf{r}_{\tilde{u}} + \tilde{\Gamma}^2_{11}\mathbf{r}_{\tilde{v}} + \tilde{b}_{11}\tilde{\mathbf{n}} \\
\mathbf{r}_{\tilde{u}\tilde{v}} &= \tilde{\Gamma}^1_{12}\mathbf{r}_{\tilde{u}} + \tilde{\Gamma}^2_{12}\mathbf{r}_{\tilde{v}} + \tilde{b}_{12}\tilde{\mathbf{n}} \\
\mathbf{r}_{\tilde{v}\tilde{u}} &= \tilde{\Gamma}^1_{21}\mathbf{r}_{\tilde{u}} + \tilde{\Gamma}^2_{21}\mathbf{r}_{\tilde{v}} + \tilde{b}_{21}\tilde{\mathbf{n}} \\
\mathbf{r}_{\tilde{v}\tilde{v}} &= \tilde{\Gamma}^1_{22}\mathbf{r}_{\tilde{u}} + \tilde{\Gamma}^2_{22}\mathbf{r}_{\tilde{v}} + \tilde{b}_{22}\tilde{\mathbf{n}}.
\end{aligned} \tag{40}$$

Differentiating the basis transformation given in Eq. 12 with respect to the coordinates (u,v), we also have

$$\begin{aligned}
\mathbf{r}_{uu} &= (\mathbf{r}_{\tilde{u}\tilde{u}}\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{u}\tilde{v}}\frac{\partial\tilde{v}}{\partial u})\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{u}}\frac{\partial^2\tilde{u}}{\partial u\partial u} \\
&+ (\mathbf{r}_{\tilde{v}\tilde{u}}\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{v}\tilde{v}}\frac{\partial\tilde{v}}{\partial u})\frac{\partial\tilde{v}}{\partial u} + \mathbf{r}_{\tilde{v}}\frac{\partial^2\tilde{v}}{\partial u\partial u} \\
\mathbf{r}_{uv} &= (\mathbf{r}_{\tilde{u}\tilde{u}}\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{u}\tilde{v}}\frac{\partial\tilde{v}}{\partial v})\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{u}}\frac{\partial^2\tilde{u}}{\partial v\partial u} \\
&+ (\mathbf{r}_{\tilde{v}\tilde{u}}\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{v}\tilde{v}}\frac{\partial\tilde{v}}{\partial v})\frac{\partial\tilde{v}}{\partial u} + \mathbf{r}_{\tilde{v}}\frac{\partial^2\tilde{v}}{\partial v\partial u} \\
\mathbf{r}_{vu} &= (\mathbf{r}_{\tilde{u}\tilde{u}}\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{u}\tilde{v}}\frac{\partial\tilde{v}}{\partial u})\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{u}}\frac{\partial^2\tilde{u}}{\partial u\partial v} \\
&+ (\mathbf{r}_{\tilde{v}\tilde{u}}\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{v}\tilde{v}}\frac{\partial\tilde{v}}{\partial v})\frac{\partial\tilde{v}}{\partial u} + \mathbf{r}_{\tilde{v}}\frac{\partial^2\tilde{v}}{\partial v\partial u} \\
\mathbf{r}_{vv} &= (\mathbf{r}_{\tilde{u}\tilde{u}}\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{u}\tilde{v}}\frac{\partial\tilde{v}}{\partial v})\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{u}}\frac{\partial^2\tilde{u}}{\partial v\partial v} \\
&+ (\mathbf{r}_{\tilde{v}\tilde{u}}\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{v}\tilde{v}}\frac{\partial\tilde{v}}{\partial v})\frac{\partial\tilde{v}}{\partial v} + \mathbf{r}_{\tilde{v}}\frac{\partial^2\tilde{v}}{\partial v\partial v}.
\end{aligned} \tag{41}$$

We can further equate the terms on the right-hand sides of Eq. 41 and Eq. 40

$$\begin{aligned}
\Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + b_{11}\mathbf{n} &= (\tilde{\Gamma}^1_{11}\mathbf{r}_{\tilde{u}} + \tilde{\Gamma}^2_{11}\mathbf{r}_{\tilde{v}} + \tilde{b}_{11}\tilde{\mathbf{n}})\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + (\tilde{\Gamma}^1_{12}\mathbf{r}_{\tilde{u}} + \tilde{\Gamma}^2_{12}\mathbf{r}_{\tilde{v}} + \tilde{b}_{12}\tilde{\mathbf{n}})\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{u}}\frac{\partial^2\tilde{u}}{\partial u\partial u} \\
&+(\tilde{\Gamma}^1_{21}\mathbf{r}_{\tilde{u}} + \tilde{\Gamma}^2_{21}\mathbf{r}_{\tilde{v}} + \tilde{b}_{21}\tilde{\mathbf{n}})\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + (\tilde{\Gamma}^1_{22}\mathbf{r}_{\tilde{u}} + \tilde{\Gamma}^2_{22}\mathbf{r}_{\tilde{v}} + \tilde{b}_{22}\tilde{\mathbf{n}})\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \mathbf{r}_{\tilde{v}}\frac{\partial^2\tilde{v}}{\partial u\partial u} \\
&= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{u}}{\partial u\partial u})\mathbf{r}_{\tilde{u}} \\
&+(\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{v}}{\partial u\partial u})\mathbf{r}_{\tilde{v}} \\
&+(\tilde{b}_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{b}_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{b}_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{b}_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u})\tilde{\mathbf{n}} \\
\Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + b_{12}\mathbf{n} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{u}}{\partial v\partial u})\mathbf{r}_{\tilde{u}} \\
&+(\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{v}}{\partial v\partial u})\mathbf{r}_{\tilde{v}} \\
&+(\tilde{b}_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u} + \tilde{b}_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{b}_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{b}_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u})\tilde{\mathbf{n}} \\
\Gamma^1_{21}\mathbf{r}_u + \Gamma^2_{21}\mathbf{r}_v + b_{21}\mathbf{n} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{u}}{\partial u\partial v})\mathbf{r}_{\tilde{u}} \\
&+(\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{v}}{\partial u\partial v})\mathbf{r}_{\tilde{v}} \\
&+(\tilde{b}_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{b}_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{b}_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{b}_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v})\tilde{\mathbf{n}}.
\end{aligned}$$

$$
\begin{aligned}
\Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + b_{22}\mathbf{n} \quad = \quad & (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{u}}{\partial v\partial v})\mathbf{r}_{\tilde{u}} \\
& +(\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{v}}{\partial v\partial v})\mathbf{r}_{\tilde{v}} \\
& +(\tilde{b}_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v} + \tilde{b}_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{b}_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{b}_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v})\tilde{\mathbf{n}}.
\end{aligned}
$$

As $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{r}_{\tilde{u}}$ and $\mathbf{r}_{\tilde{v}}$ are perpedicular to $\mathbf{n}$, the following expressions should be satisfied

$$
\begin{aligned}
\Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v &= \Gamma^1_{11}(\mathbf{r}_{\tilde{u}}\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{v}}\frac{\partial\tilde{v}}{\partial u}) + \Gamma^2_{11}(\mathbf{r}_{\tilde{u}}\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{v}}\frac{\partial\tilde{v}}{\partial v}) \\
&= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{u}}{\partial u\partial u})\mathbf{r}_{\tilde{u}} \\
&\quad +(\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{v}}{\partial u\partial u})\mathbf{r}_{\tilde{v}} \\
\Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v &= \Gamma^1_{12}(\mathbf{r}_{\tilde{u}}\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{v}}\frac{\partial\tilde{v}}{\partial u}) + \Gamma^2_{12}(\mathbf{r}_{\tilde{u}}\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{v}}\frac{\partial\tilde{v}}{\partial v}) \\
&= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{u}}{\partial v\partial u})\mathbf{r}_{\tilde{u}} \\
&\quad +(\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{v}}{\partial v\partial u})\mathbf{r}_{\tilde{v}} \\
\Gamma^1_{21}\mathbf{r}_u + \Gamma^2_{21}\mathbf{r}_v &= \Gamma^1_{21}(\mathbf{r}_{\tilde{u}}\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{v}}\frac{\partial\tilde{v}}{\partial u}) + \Gamma^2_{21}(\mathbf{r}_{\tilde{u}}\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{v}}\frac{\partial\tilde{v}}{\partial v}) \\
&= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{u}}{\partial u\partial v})\mathbf{r}_{\tilde{u}} \\
&\quad +(\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{v}}{\partial u\partial v})\mathbf{r}_{\tilde{v}} \\
\Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v &= \Gamma^1_{22}(\mathbf{r}_{\tilde{u}}\frac{\partial\tilde{u}}{\partial u} + \mathbf{r}_{\tilde{v}}\frac{\partial\tilde{v}}{\partial u}) + \Gamma^2_{22}(\mathbf{r}_{\tilde{u}}\frac{\partial\tilde{u}}{\partial v} + \mathbf{r}_{\tilde{v}}\frac{\partial\tilde{v}}{\partial v}) \\
&= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{u}}{\partial v\partial v})\mathbf{r}_{\tilde{u}} \\
&\quad +(\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{v}}{\partial v\partial v})\mathbf{r}_{\tilde{v}}.
\end{aligned}
$$

Observe that Eq. 12 has been applied to get formulations most suitable for our purpose: to equate the coefficients of the basis vectors on the left-hand and on the right-hand sides. Moreover, we recall that the parameters u and v are independent, as are the parameters $\tilde{u}$ and $\tilde{v}$. We thus find the following *formulae of Christoffel*

$$
\begin{aligned}
(\Gamma^1_{11}\frac{\partial\tilde{u}}{\partial u} + \Gamma^2_{11}\frac{\partial\tilde{u}}{\partial v})\frac{\partial u}{\partial\tilde{u}} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{u}}{\partial u\partial u})\frac{\partial u}{\partial\tilde{u}} \\
\Gamma^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial u}{\partial\tilde{u}} = \Gamma^1_{11} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{u}}{\partial u\partial u})\frac{\partial u}{\partial\tilde{u}} \\
(\Gamma^1_{11}\frac{\partial\tilde{u}}{\partial u} + \Gamma^2_{11}\frac{\partial\tilde{u}}{\partial v})\frac{\partial v}{\partial\tilde{u}} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{u}}{\partial u\partial u})\frac{\partial v}{\partial\tilde{u}} \\
\Gamma^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial v}{\partial\tilde{u}} = \Gamma^2_{11} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{u}}{\partial u\partial u})\frac{\partial v}{\partial\tilde{u}} \\
(\Gamma^1_{11}\frac{\partial\tilde{v}}{\partial u} + \Gamma^2_{11}\frac{\partial\tilde{v}}{\partial v})\frac{\partial u}{\partial\tilde{v}} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{v}}{\partial u\partial u})\frac{\partial u}{\partial\tilde{v}} \\
\Gamma^1_{11}\frac{\partial\tilde{v}}{\partial u}\frac{\partial u}{\partial\tilde{v}} = \Gamma^1_{11} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{v}}{\partial u\partial u})\frac{\partial u}{\partial\tilde{v}} \\
(\Gamma^1_{11}\frac{\partial\tilde{v}}{\partial u} + \Gamma^2_{11}\frac{\partial\tilde{v}}{\partial v})\frac{\partial v}{\partial\tilde{v}} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{v}}{\partial u\partial u})\frac{\partial v}{\partial\tilde{v}} \\
\Gamma^2_{11}\frac{\partial\tilde{v}}{\partial v}\frac{\partial v}{\partial\tilde{v}} = \Gamma^2_{11} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \frac{\partial^2\tilde{v}}{\partial u\partial u})\frac{\partial v}{\partial\tilde{v}} \\
(\Gamma^1_{12}\frac{\partial\tilde{u}}{\partial u} + \Gamma^2_{12}\frac{\partial\tilde{u}}{\partial v})\frac{\partial u}{\partial\tilde{u}} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{u}}{\partial u\partial v})\frac{\partial u}{\partial\tilde{u}} \\
\Gamma^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial u}{\partial\tilde{u}} = \Gamma^1_{12} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \frac{\partial^2\tilde{u}}{\partial u\partial v})\frac{\partial u}{\partial\tilde{u}}
\end{aligned}
$$

$$
\begin{aligned}
(\Gamma^1_{12}\frac{\partial\tilde{u}}{\partial u}+\Gamma^2_{12}\frac{\partial\tilde{u}}{\partial v})\frac{\partial v}{\partial\tilde{u}} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{u}}{\partial u\partial v})\frac{\partial v}{\partial\tilde{u}}\\
\Gamma^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial v}{\partial\tilde{u}}=\Gamma^2_{12} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{u}}{\partial u\partial v})\frac{\partial v}{\partial\tilde{u}}\\
(\Gamma^1_{12}\frac{\partial\tilde{v}}{\partial u}+\Gamma^2_{12}\frac{\partial\tilde{v}}{\partial v})\frac{\partial u}{\partial\tilde{v}} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{v}}{\partial u\partial v})\frac{\partial u}{\partial\tilde{v}}\\
\Gamma^1_{12}\frac{\partial\tilde{v}}{\partial u}\frac{\partial u}{\partial\tilde{v}}=\Gamma^1_{12} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{v}}{\partial u\partial v})\frac{\partial u}{\partial\tilde{v}}\\
(\Gamma^1_{12}\frac{\partial\tilde{v}}{\partial u}+\Gamma^2_{12}\frac{\partial\tilde{v}}{\partial v})\frac{\partial v}{\partial\tilde{v}} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{v}}{\partial u\partial v})\frac{\partial v}{\partial\tilde{v}}\\
\Gamma^2_{12}\frac{\partial\tilde{v}}{\partial v}\frac{\partial v}{\partial\tilde{v}}=\Gamma^2_{12} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{v}}{\partial u\partial v})\frac{\partial v}{\partial\tilde{v}}\\
(\Gamma^1_{21}\frac{\partial\tilde{u}}{\partial u}+\Gamma^2_{21}\frac{\partial\tilde{u}}{\partial v})\frac{\partial u}{\partial\tilde{u}} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\frac{\partial^2\tilde{u}}{\partial v\partial u})\frac{\partial u}{\partial\tilde{u}}\\
\Gamma^1_{21}\frac{\partial\tilde{u}}{\partial u}\frac{\partial u}{\partial\tilde{u}}=\Gamma^1_{21} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\frac{\partial^2\tilde{u}}{\partial v\partial u})\frac{\partial u}{\partial\tilde{u}}\\
(\Gamma^1_{21}\frac{\partial\tilde{u}}{\partial u}+\Gamma^2_{21}\frac{\partial\tilde{u}}{\partial v})\frac{\partial v}{\partial\tilde{u}} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\frac{\partial^2\tilde{u}}{\partial v\partial u})\frac{\partial v}{\partial\tilde{u}}\\
\Gamma^2_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial v}{\partial\tilde{u}}=\Gamma^2_{21} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\frac{\partial^2\tilde{u}}{\partial v\partial u})\frac{\partial v}{\partial\tilde{u}}\\
(\Gamma^1_{21}\frac{\partial\tilde{u}}{\partial v}+\Gamma^2_{21}\frac{\partial\tilde{v}}{\partial v})\frac{\partial v}{\partial\tilde{u}} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\frac{\partial^2\tilde{v}}{\partial v\partial u})\frac{\partial v}{\partial\tilde{u}}\\
\Gamma^1_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial v}{\partial\tilde{u}}=\Gamma^1_{21} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\frac{\partial^2\tilde{v}}{\partial v\partial u})\frac{\partial v}{\partial\tilde{u}}\\
(\Gamma^1_{21}\frac{\partial\tilde{u}}{\partial v}+\Gamma^2_{21}\frac{\partial\tilde{v}}{\partial v})\frac{\partial v}{\partial\tilde{v}} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\frac{\partial^2\tilde{v}}{\partial v\partial u})\frac{\partial v}{\partial\tilde{v}}\\
\Gamma^2_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial v}{\partial\tilde{v}}=\Gamma^2_{21} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial u}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial u}+\frac{\partial^2\tilde{v}}{\partial v\partial u})\frac{\partial v}{\partial\tilde{v}}\\
(\Gamma^1_{22}\frac{\partial\tilde{u}}{\partial u}+\Gamma^2_{22}\frac{\partial\tilde{u}}{\partial v})\frac{\partial u}{\partial\tilde{u}} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{u}}{\partial v\partial v})\frac{\partial u}{\partial\tilde{u}}\\
\Gamma^1_{22}\frac{\partial\tilde{u}}{\partial u}\frac{\partial u}{\partial\tilde{u}}=\Gamma^1_{22} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{u}}{\partial v\partial v})\frac{\partial u}{\partial\tilde{u}}\\
(\Gamma^1_{22}\frac{\partial\tilde{u}}{\partial u}+\Gamma^2_{22}\frac{\partial\tilde{u}}{\partial v})\frac{\partial v}{\partial\tilde{u}} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{u}}{\partial v\partial v})\frac{\partial v}{\partial\tilde{u}}\\
\Gamma^2_{22}\frac{\partial\tilde{u}}{\partial v}\frac{\partial v}{\partial\tilde{u}}=\Gamma^2_{22} &= (\tilde{\Gamma}^1_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^1_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^1_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{u}}{\partial v\partial v})\frac{\partial v}{\partial\tilde{u}}\\
(\Gamma^1_{22}\frac{\partial\tilde{v}}{\partial u}+\Gamma^2_{22}\frac{\partial\tilde{v}}{\partial v})\frac{\partial u}{\partial\tilde{v}} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{v}}{\partial v\partial v})\frac{\partial u}{\partial\tilde{v}}\\
\Gamma^1_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial u}{\partial\tilde{v}}=\Gamma^1_{22} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{v}}{\partial v\partial v})\frac{\partial u}{\partial\tilde{v}}\\
(\Gamma^1_{22}\frac{\partial\tilde{v}}{\partial u}+\Gamma^2_{22}\frac{\partial\tilde{v}}{\partial v})\frac{\partial v}{\partial\tilde{v}} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{v}}{\partial v\partial v})\frac{\partial v}{\partial\tilde{v}}\\
\Gamma^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial v}{\partial\tilde{v}}=\Gamma^2_{22} &= (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{v}}{\partial v\partial v})\frac{\partial v}{\partial\tilde{v}}.
\end{aligned}
$$

The transformation law of the Christoffel symbols of the first kind follows immediately from Eq. 32 and Eq. 15. For example,

$$
\begin{aligned}
\Gamma_{122} &= a_{21}\Gamma^1_{12}+a_{22}\Gamma^2_{12}=a_{12}\Gamma^1_{12}+a_{22}\Gamma^2_{12}\\
&= (\tilde{a}_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{a}_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\tilde{a}_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{a}_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v})\Gamma^1_{12}+\\
&\quad +(\tilde{a}_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{a}_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v}+\tilde{a}_{21}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{u}}{\partial v}+\tilde{a}_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v})\Gamma^2_{12}\\
&= (\tilde{a}_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{a}_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\tilde{a}_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{a}_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v})(\frac{\partial u}{\partial\tilde{v}}\\
&\quad (\tilde{\Gamma}^2_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\tilde{\Gamma}^2_{21}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\tilde{\Gamma}^2_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v}+\frac{\partial^2\tilde{v}}{\partial u\partial v}))+
\end{aligned}
$$

$$
\begin{aligned}
& +(\tilde a_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial v}+\tilde a_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v}+\tilde a_{21}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde u}{\partial v}+\tilde a_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial v})(\frac{\partial v}{\partial\tilde u} \\
& (\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}+\tilde\Gamma^1_{21}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v}+\frac{\partial^2\tilde u}{\partial u\partial v})) \\
= \; & (\tilde a_{21}\frac{\partial\tilde u}{\partial v}+\tilde a_{22}\frac{\partial\tilde v}{\partial v}) \\
& (\tilde\Gamma^2_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma^2_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}+\tilde\Gamma^2_{21}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma^2_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v}+\frac{\partial^2\tilde v}{\partial u\partial v})+ \\
& +(\tilde a_{11}\frac{\partial\tilde u}{\partial v}+\tilde a_{12}\frac{\partial\tilde v}{\partial v}) \\
& (\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}+\tilde\Gamma^1_{21}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v}+\frac{\partial^2\tilde u}{\partial u\partial v}) \\
= \; & \frac{\partial\tilde u}{\partial v}((\tilde a_{21}\tilde\Gamma^2_{11}+\tilde a_{11}\tilde\Gamma^1_{11})\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}+(\tilde a_{21}\tilde\Gamma^2_{12}+\tilde a_{11}\tilde\Gamma^1_{12})\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}+(\tilde a_{21}\tilde\Gamma^2_{21}+\tilde a_{11}\tilde\Gamma^1_{21})\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde u}{\partial v}+ \\
& +(\tilde a_{21}\tilde\Gamma^2_{22}+\tilde a_{11}\tilde\Gamma^1_{22})\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v}+(\tilde a_{11}\frac{\partial^2\tilde u}{\partial u\partial v}+\tilde a_{21}\frac{\partial^2\tilde v}{\partial u\partial v}))+ \\
& +\frac{\partial\tilde v}{\partial v}((\tilde a_{22}\tilde\Gamma^2_{11}+\tilde a_{12}\tilde\Gamma^1_{11})\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}+(\tilde a_{22}\tilde\Gamma^2_{12}+\tilde a_{12}\tilde\Gamma^1_{12})\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}+(\tilde a_{22}\tilde\Gamma^2_{21}+\tilde a_{12}\tilde\Gamma^1_{21})\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde u}{\partial v}+ \\
& +(\tilde a_{22}\tilde\Gamma^2_{22}+\tilde a_{12}\tilde\Gamma^1_{22})\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v}+(\tilde a_{12}\frac{\partial^2\tilde u}{\partial u\partial v}+\tilde a_{22}\frac{\partial^2\tilde v}{\partial u\partial v})) \\
= \; & \frac{\partial\tilde u}{\partial v}(\tilde\Gamma_{111}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma_{121}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}+\tilde\Gamma_{211}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma_{221}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v}+(\tilde a_{11}\frac{\partial^2\tilde u}{\partial u\partial v}+\tilde a_{21}\frac{\partial^2\tilde v}{\partial u\partial v}))+ \\
& +\frac{\partial\tilde v}{\partial v}(\tilde\Gamma_{112}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma_{122}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}+\tilde\Gamma_{212}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde u}{\partial v}+\tilde\Gamma_{222}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v}+(\tilde a_{12}\frac{\partial^2\tilde u}{\partial u\partial v}+\tilde a_{22}\frac{\partial^2\tilde v}{\partial u\partial v}))
\end{aligned}
$$

From this transformation law we may further derive the second derivatives of the coordinates $(\tilde u, \tilde v)$ with respect to the coordinates (u, v). Observe that they only depend on the derivatives of first order and the Christoffel symbols

$$
\begin{aligned}
\frac{\partial^2\tilde u}{\partial u\partial u} &= \Gamma^1_{11}\frac{\partial\tilde u}{\partial u}+\Gamma^2_{11}\frac{\partial\tilde u}{\partial v}-\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial u}-\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u}-\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u}-\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial u} \\
\frac{\partial^2\tilde v}{\partial u\partial u} &= \Gamma^1_{11}\frac{\partial\tilde v}{\partial u}+\Gamma^2_{11}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^2_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial u}-\tilde\Gamma^2_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u}-\tilde\Gamma^2_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u}-\tilde\Gamma^2_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial u} \\
\frac{\partial^2\tilde u}{\partial u\partial v} &= \Gamma^1_{12}\frac{\partial\tilde u}{\partial u}+\Gamma^2_{12}\frac{\partial\tilde u}{\partial v}-\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}-\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v} \\
\frac{\partial^2\tilde v}{\partial u\partial v} &= \Gamma^1_{12}\frac{\partial\tilde v}{\partial u}+\Gamma^2_{12}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}-\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v} \\
\frac{\partial^2\tilde u}{\partial v\partial u} &= \Gamma^1_{21}\frac{\partial\tilde u}{\partial u}+\Gamma^2_{21}\frac{\partial\tilde u}{\partial v}-\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial u}-\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u}-\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u}-\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial u} \\
\frac{\partial^2\tilde v}{\partial v\partial u} &= \Gamma^1_{21}\frac{\partial\tilde u}{\partial v}+\Gamma^2_{21}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial u}-\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u}-\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u}-\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial u} \\
\frac{\partial^2\tilde u}{\partial v\partial v} &= \Gamma^1_{22}\frac{\partial\tilde u}{\partial u}+\Gamma^2_{22}\frac{\partial\tilde u}{\partial v}-\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial v}-\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial v} \\
\frac{\partial^2\tilde v}{\partial v\partial v} &= \Gamma^1_{22}\frac{\partial\tilde v}{\partial u}+\Gamma^2_{22}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial v}-\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v}-\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial v}
\end{aligned}
\tag{42}
$$

2.4.3 Curvature Tensors

The *unit normal vector* of the tangent plane expressed in Eq. 8 is given by

$$
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{a_{11}a_{22} - a_{12}^2}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{a}}. \tag{43}
$$

The straight line through $\mathcal{P}$ in the direction of $\mathbf{n}$ is called the *normal* to the surface $\mathcal{S}$ at the point $\mathcal{P}$. We may, hence, define a local reference consisting $\mathbf{r}_u$, $\mathbf{r}_v$ and $\mathbf{n}$ to $\mathcal{P}$, whose lengths are, respectively, a_{11}, a_{22}, and 1.

Now, let us compute the variations of the tangent vector at $\mathcal{P}$ in a direction $\alpha(t)$

$$
\frac{d^2\mathbf{r}}{dt} = (\frac{\partial^2\mathbf{r}}{\partial u\partial u}\frac{du}{dt} + \frac{\partial^2\mathbf{r}}{\partial v\partial u}\frac{dv}{dt})\frac{du}{dt} + \mathbf{r}_u\frac{d^2u}{d^2t} + (\frac{\partial^2\mathbf{r}}{\partial u\partial v}\frac{du}{dt} + \frac{\partial^2\mathbf{r}}{\partial v\partial v}\frac{dv}{dt})\frac{dv}{dt} + \mathbf{r}_v\frac{d^2v}{d^2t},
$$

take the dot product of $\mathbf{n}$ and both sides of the expression

$$
\begin{aligned}
\frac{d^2\mathbf{r}}{dt}\cdot\mathbf{n} &= (\frac{\partial^2\mathbf{r}}{\partial u\partial u}\frac{du}{dt}+\frac{\partial^2\mathbf{r}}{\partial v\partial u}\frac{dv}{dt})\frac{du}{dt}\cdot\mathbf{n}+\mathbf{r}_u\frac{d^2u}{d^2t}\cdot\mathbf{n} \\
&\quad +(\frac{\partial^2\mathbf{r}}{\partial u\partial v}\frac{du}{dt}\cdot\mathbf{n}+\frac{\partial^2\mathbf{r}}{\partial v\partial v}\frac{dv}{dt})\frac{dv}{dt}\cdot\mathbf{n}+\mathbf{r}_v\frac{d^2v}{d^2t}\cdot\mathbf{n} \\
&= (\frac{\partial^2\mathbf{r}}{\partial u\partial u}\cdot\mathbf{n})\frac{du}{dt}\frac{du}{dt}+(\frac{\partial^2\mathbf{r}}{\partial v\partial u}\cdot\mathbf{n})\frac{dv}{dt}\frac{du}{dt} \\
&\quad +(\frac{\partial^2\mathbf{r}}{\partial u\partial v}\cdot\mathbf{n})\frac{du}{dt}\frac{dv}{dt}+(\frac{\partial^2\mathbf{r}}{\partial v\partial v}\cdot\mathbf{n})\frac{dv}{dt}\frac{dv}{dt},
\end{aligned}
\tag{44}
$$

and denote, respectively, the terms $(\frac{\partial^2\mathbf{r}}{\partial u\partial u}\cdot\mathbf{n})$, $(\frac{\partial^2\mathbf{r}}{\partial u\partial v}\cdot\mathbf{n})$, $(\frac{\partial^2\mathbf{r}}{\partial v\partial u}\cdot\mathbf{n})$ and $(\frac{\partial^2\mathbf{r}}{\partial v\partial v}\cdot\mathbf{n})$ as b_{11}, b_{12}, b_{21} and b_{22}, we reach an expression similar to Eq. 9

$$
\begin{aligned}
\kappa_n = II(d^2\alpha(t),\mathbf{n}) &= \frac{d^2\mathbf{r}}{dt}\cdot\mathbf{n} \equiv b_{11}\frac{du}{dt}\frac{du}{dt}+b_{12}\frac{du}{dt}\frac{dv}{dt}+b_{21}\frac{du}{dt}\frac{dv}{dt}+b_{22}\frac{dv}{dt}\frac{dv}{dt} \\
&\equiv \begin{bmatrix}\frac{du}{dt} & \frac{dv}{dt}\end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}\frac{du}{dt} \\ \frac{dv}{dt}\end{bmatrix},
\end{aligned}
\tag{45}
$$

which is called the *second fundamental form*. The quantity $\frac{d^2\mathbf{r}}{dt}\cdot\mathbf{n}$ is also known the *normal curvature*, which possesses a geometric meaning independent of the choice of the coordinates. It is the curvature of the normal section of $\mathcal{S}$ along the direction $d\mathbf{r} = \alpha(t) = \frac{\partial\mathbf{r}}{\partial u}\frac{du}{dt}+\frac{\partial\mathbf{r}}{\partial v}\frac{dv}{dt}$[1] on the tangent plane. Since the coordinate differentiations $\frac{du}{dt}$ and $\frac{dv}{dt}$ have a contravariant transformation (Eq. 17) and $II(d^2\alpha(t),\mathbf{n})$ is invariant, the components b_{11}, b_{12}, b_{21}, and b_{22} form a covariant tensor of second order called *covariant curvature tensor* and they are known as *curvature coefficients*

Differentiating the orthogonality relations $\mathbf{r}_u\cdot\mathbf{n}=0$ and $\mathbf{r}_v\cdot\mathbf{n}=0$, we have

$$
\begin{aligned}
\frac{\partial^2\mathbf{r}}{\partial u\partial u}\cdot\mathbf{n}+\mathbf{r}_u\cdot\frac{\partial\mathbf{n}}{\partial u}=0 &\implies \frac{\partial^2\mathbf{r}}{\partial u\partial u}\cdot\mathbf{n}=-\mathbf{r}_u\cdot\frac{\partial\mathbf{n}}{\partial u} \\
\frac{\partial^2\mathbf{r}}{\partial u\partial v}\cdot\mathbf{n}+\mathbf{r}_v\cdot\frac{\partial\mathbf{n}}{\partial u}=0 &\implies \frac{\partial^2\mathbf{r}}{\partial u\partial v}\cdot\mathbf{n}=-\mathbf{r}_v\cdot\frac{\partial\mathbf{n}}{\partial u} \\
\frac{\partial^2\mathbf{r}}{\partial v\partial u}\cdot\mathbf{n}+\mathbf{r}_u\cdot\frac{\partial\mathbf{n}}{\partial v}=0 &\implies \frac{\partial^2\mathbf{r}}{\partial v\partial u}\cdot\mathbf{n}=-\mathbf{r}_u\cdot\frac{\partial\mathbf{n}}{\partial v} \\
\frac{\partial^2\mathbf{r}}{\partial v\partial v}\cdot\mathbf{n}+\mathbf{r}_v\cdot\frac{\partial\mathbf{n}}{\partial v}=0 &\implies \frac{\partial^2\mathbf{r}}{\partial v\partial v}\cdot\mathbf{n}=-\mathbf{r}_v\cdot\frac{\partial\mathbf{n}}{\partial v}.
\end{aligned}
$$

This leads us to important equalities that relate the orthogonal projections of the derivatives of $\mathbf{n}$ with the coefficients of the second fundamental form

$$
\begin{aligned}
b_{11} &= \frac{\partial^2\mathbf{r}}{\partial u\partial u}\cdot\mathbf{n}=-\mathbf{r}_u\cdot\frac{\partial\mathbf{n}}{\partial u} \\
b_{12} &= \frac{\partial^2\mathbf{r}}{\partial u\partial v}\cdot\mathbf{n}=-\mathbf{r}_u\cdot\frac{\partial\mathbf{n}}{\partial v} \\
b_{21} &= \frac{\partial^2\mathbf{r}}{\partial v\partial u}\cdot\mathbf{n}=-\mathbf{r}_v\cdot\frac{\partial\mathbf{n}}{\partial u} \\
b_{11} &= \frac{\partial^2\mathbf{r}}{\partial v\partial v}\cdot\mathbf{n}=-\mathbf{r}_v\cdot\frac{\partial\mathbf{n}}{\partial v}
\end{aligned}
\tag{46}
$$

and an alternative geometrical interpretation to the second fundamental form

$$
II(\alpha(t),d_{\alpha(t)}\mathbf{n}) = -d\mathbf{r}\cdot d_{\alpha(t)}\mathbf{n} = b_{11}\frac{du}{dt}\frac{du}{dt}+b_{12}\frac{du}{dt}\frac{dv}{dt}+b_{21}\frac{du}{dt}\frac{dv}{dt}+b_{22}\frac{dv}{dt}\frac{dv}{dt} \tag{47}
$$

The term $-d_{\alpha(t)}\mathbf{n}$ gives, indeed, the "shape" of $\mathcal{S}$ in the vicinity of $\mathcal{P}$ along the direction $\alpha(t)$ on the tangent plane. It is also called the *shape operator*

$$
S(\alpha(t)) = -d_{\alpha(t)}\mathbf{n}.
$$

[1] The normal section is the curve of intersection of the surface $\mathcal{S}$ and a plane that passes through both the tangent $d\mathbf{r}$ to the curve at the point $\mathcal{P}$ and the normal to $\mathcal{S}$

The shape operator of $\mathcal{S}$ may be expressed as a linear combination of the basis vectors $(\mathbf{r}_u, \mathbf{r}_v)$

$$\begin{bmatrix} -S(\mathbf{r}_u) \\ -S(\mathbf{r}_v) \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{n}}{\partial u} \\ \frac{\partial \mathbf{n}}{\partial v} \end{bmatrix} = \begin{bmatrix} -b_1^1 & -b_1^2 \\ -b_2^1 & -b_2^2 \end{bmatrix} \begin{bmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{bmatrix} \tag{48}$$

This follows from the fact that $\mathbf{n} \cdot \mathbf{n} = 1$ and we have $\mathbf{n}_u \cdot \mathbf{n} = 0$ and $\mathbf{n}_v \cdot \mathbf{n} = 0$. And with use of this operator we may write the second fundamental form as follows

$$II(\alpha(t), d_{\alpha(t)}\mathbf{n}) = S(\alpha(t)).d\mathbf{r}. \tag{49}$$

Taking the dot product of both sides of Eq. 48 and the derivatives of $\mathbf{r}$, we find

$$\begin{aligned}
\frac{\partial \mathbf{n}}{\partial u} \cdot \mathbf{r}_u = -b_{11} &= -b_1^1 \mathbf{r}_u \cdot \mathbf{r}_u - b_2^1 \mathbf{r}_v \cdot \mathbf{r}_u = -b_1^1 a_{11} - b_2^1 a_{21} \\
\frac{\partial \mathbf{n}}{\partial u} \cdot \mathbf{r}_v = -b_{12} &= -b_1^1 \mathbf{r}_u \cdot \mathbf{r}_v - b_2^1 \mathbf{r}_v \cdot \mathbf{r}_v = -b_1^1 a_{12} - b_2^1 a_{22} \\
\frac{\partial \mathbf{n}}{\partial v} \cdot \mathbf{r}_u = -b_{21} &= -b_1^2 \mathbf{r}_u \cdot \mathbf{r}_u - b_2^2 \mathbf{r}_v \cdot \mathbf{r}_u = -b_1^2 a_{11} - b_2^2 a_{21} \\
\frac{\partial \mathbf{n}}{\partial v} \cdot \mathbf{r}_v = -b_{22} &= -b_1^2 \mathbf{r}_u \cdot \mathbf{r}_v - b_2^2 \mathbf{r}_v \cdot \mathbf{r}_v = -b_1^2 a_{12} - b_2^2 a_{22}
\end{aligned}$$

and, after solving the linear equation system, we obtain the Weingarten equations

$$\begin{bmatrix} \mathbf{n}_u \\ \mathbf{n}_v \end{bmatrix} = \begin{bmatrix} -b_1^1 & -b_2^1 \\ -b_1^2 & -b_2^2 \end{bmatrix} \begin{bmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{bmatrix} = \begin{bmatrix} \frac{b_{12}a_{12} - b_{11}a_{22}}{a_{11}a_{22} - a_{12}^2} & \frac{b_{11}a_{12} - b_{12}a_{11}}{a_{11}a_{22} - a_{12}^2} \\ \frac{b_{22}a_{12} - b_{12}a_{22}}{a_{11}a_{22} - a_{12}^2} & \frac{b_{12}a_{12} - b_{22}a_{11}}{a_{11}a_{22} - a_{12}^2} \end{bmatrix} \begin{bmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{bmatrix} \tag{50}$$

Although there is an infinity of curves on $\mathcal{S}$ passing through $\mathcal{P}$ tangent to the same direction, the theorem of Meusnier tells us that we may restrict our consideration only to the normal section of $\mathcal{S}$ on that direction, once all of those curves have the same normal curvature. In order to determine the shape of $\mathcal{S}$ in the vicinity of any of its points, we should still investigate the normal curvature of all the normal sections at $\mathcal{P}$. The direction of the normal curvature always remains that of the surface normal, but its length may vary for different directions. The directions in which the normal curvature becomes extreme are called *principal directions* and the corresponding normal curvatures are denominated *principal curvatures*, κ_1 and κ_2. These principal directions and curvatures are the eigenvectors and the eigenvalues of the matrix b_α^β in Eq. 50.

If in particular the coordinates are chosen so that the coordinate curves are lines of curvature on the surface $\mathcal{S}$, then the following properties are also satisfied

$$a_{12} = b_{12} = 0 \tag{51}$$

$$\kappa_1 = \frac{b_{11}}{a_{11}}$$

$$\kappa_2 = \frac{b_{22}}{a_{22}} \tag{52}$$

If a_{11} and a_{22} are furthermore unitary the principal curvatures coincide with the curvature coefficients.

2.4.4 Covariant Differentiation

The ordinary derivative of a vector field is defined in terms of the difference between two vectors at two nearby points. In the Cartesian coordinate system we simply translate one of the vectors to the origin of the other, keeping it parallel. If the coordinates are chosen so that the coordinates are curvilinear lines on the surface $\mathcal{S}$, this translation is not well defined because the coordinate transformation given in Eq. 11 is not linear. Depending on the closed path along which the vector is displaced, it may not return as the same vector as depicted in Figure 5. We must define a rule that makes the vector displacements unique and, thus, comparable. The covariant differentiation meets this requirement.

One way to consider the covariant differentiation is that it is a coordinate differentiation of a vector $\mathbf{s}$ taking into acount the change of basis vectors with respect to which it is defined. Let's show in this section how this derivative is related to the ordinary one. Consider as a local basis the tangent vectors $(\mathbf{r}_u, \mathbf{r}_v)$. According to Eq. 13, for passing to the new coordinates $\tilde{u} = \tilde{u}(u, v)$ and $\tilde{v} = \tilde{v}(u, v)$, the transformation matrix for the basis vectors is $J(\tilde{u}, \tilde{v})$. Hence, any contravariant components of the vector

$$\mathbf{s} = s^1 \mathbf{r}_u + s^2 \mathbf{r}_v = (s^1(u, v), s^2(u, v))$$

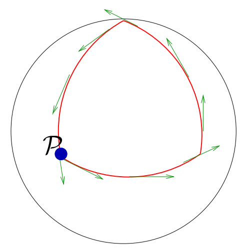

Figure 5: Displacement of a vector (in green) along a closed curvilinear path (in red).

on the tangent plane transforms in opposite direction through $J^{-1}(\tilde{u},\tilde{v})$ (Eq. 13)

$$\begin{bmatrix} \tilde{s}^1 \\ \tilde{s}^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{bmatrix} \begin{bmatrix} s^1 \\ s^2 \end{bmatrix}$$

If we take the partial derivative of the vector components with respect to the new coordinates $(\tilde{u},\tilde{v})$, we obtain the relations

$$\begin{aligned}
\frac{\partial \tilde{s}^1}{\partial \tilde{u}} &= (\frac{\partial s^1}{\partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial \tilde{u}})\frac{\partial \tilde{u}}{\partial u} + s^1(\frac{\partial^2 \tilde{u}}{\partial u \partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial^2 \tilde{u}}{\partial v \partial u}\frac{\partial v}{\partial \tilde{u}}) \\
&+ (\frac{\partial s^2}{\partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial s^2}{\partial v}\frac{\partial v}{\partial \tilde{u}})\frac{\partial \tilde{u}}{\partial v} + s^2(\frac{\partial^2 \tilde{u}}{\partial u \partial v}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial^2 \tilde{u}}{\partial v \partial v}\frac{\partial v}{\partial \tilde{u}}) \\
\frac{\partial \tilde{s}^1}{\partial \tilde{v}} &= (\frac{\partial s^1}{\partial u}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial \tilde{v}})\frac{\partial \tilde{u}}{\partial u} + s^1(\frac{\partial^2 \tilde{u}}{\partial u \partial u}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial^2 \tilde{u}}{\partial v \partial u}\frac{\partial v}{\partial \tilde{v}}) \\
&+ (\frac{\partial s^2}{\partial u}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial \tilde{v}})\frac{\partial \tilde{u}}{\partial v} + s^2(\frac{\partial^2 \tilde{u}}{\partial u \partial v}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial^2 \tilde{u}}{\partial v \partial v}\frac{\partial v}{\partial \tilde{v}}) \\
\frac{\partial \tilde{s}^2}{\partial \tilde{u}} &= (\frac{\partial s^1}{\partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial \tilde{u}})\frac{\partial \tilde{v}}{\partial u} + s^1(\frac{\partial^2 \tilde{v}}{\partial u \partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial^2 \tilde{v}}{\partial v \partial u}\frac{\partial v}{\partial \tilde{u}}) \\
&+ (\frac{\partial s^2}{\partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial s^2}{\partial v}\frac{\partial v}{\partial \tilde{u}})\frac{\partial \tilde{v}}{\partial v} + s^2(\frac{\partial^2 \tilde{v}}{\partial u \partial v}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial^2 \tilde{v}}{\partial v \partial v}\frac{\partial v}{\partial \tilde{u}}) \\
\frac{\partial \tilde{s}^2}{\partial \tilde{v}} &= (\frac{\partial s^1}{\partial u}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial \tilde{v}})\frac{\partial \tilde{v}}{\partial u} + s^1(\frac{\partial^2 \tilde{v}}{\partial u \partial u}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial^2 \tilde{v}}{\partial v \partial u}\frac{\partial v}{\partial \tilde{v}}) \\
&+ (\frac{\partial s^2}{\partial u}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial \tilde{v}})\frac{\partial \tilde{v}}{\partial v} + s^2(\frac{\partial^2 \tilde{v}}{\partial u \partial v}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial^2 \tilde{v}}{\partial v \partial v}\frac{\partial v}{\partial \tilde{v}})
\end{aligned} \tag{53}$$

and observe that on the right-hand of Eq. 53 we have the second derivatives. This means that the transformation is not in accordance with the tensor transformation law although a contravariant vector $\mathbf{s}$ is a tensor of first order. How may we rearrange the differentiation process such that the derivative of a tensor is again a tensor?

Note, however, that using Eq. 42 we may replace the second derivatives that appear in Eq. 53 and obtain

$$\begin{aligned}
\frac{\partial \tilde{s}^1}{\partial \tilde{u}} &= (\frac{\partial s^1}{\partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial \tilde{u}})\frac{\partial \tilde{u}}{\partial u} \\
&+ s^1(\Gamma^1_{11}\frac{\partial \tilde{u}}{\partial u} + \Gamma^2_{11}\frac{\partial \tilde{u}}{\partial v} - \tilde{\Gamma}^1_{11}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{u}}{\partial u} - \tilde{\Gamma}^1_{12}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial u} - \tilde{\Gamma}^1_{21}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial u} - \tilde{\Gamma}^1_{22}\frac{\partial \tilde{v}}{\partial u}\frac{\partial \tilde{v}}{\partial u})\frac{\partial u}{\partial \tilde{u}} \\
&+ s^1(\Gamma^1_{12}\frac{\partial \tilde{u}}{\partial u} + \Gamma^2_{12}\frac{\partial \tilde{u}}{\partial v} - \tilde{\Gamma}^1_{11}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{u}}{\partial u} - \tilde{\Gamma}^1_{12}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial u} - \tilde{\Gamma}^1_{21}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial u} - \tilde{\Gamma}^1_{22}\frac{\partial \tilde{v}}{\partial v}\frac{\partial \tilde{v}}{\partial u})\frac{\partial v}{\partial \tilde{u}} \\
&+ (\frac{\partial s^2}{\partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial s^2}{\partial v}\frac{\partial v}{\partial \tilde{u}})\frac{\partial \tilde{u}}{\partial v} \\
&+ s^2(\Gamma^1_{21}\frac{\partial \tilde{u}}{\partial u} + \Gamma^2_{21}\frac{\partial \tilde{u}}{\partial v} - \tilde{\Gamma}^1_{11}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{u}}{\partial v} - \tilde{\Gamma}^1_{12}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial v} - \tilde{\Gamma}^1_{21}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial v} - \tilde{\Gamma}^1_{22}\frac{\partial \tilde{v}}{\partial u}\frac{\partial \tilde{v}}{\partial v})\frac{\partial u}{\partial \tilde{u}} \\
&+ s^2(\Gamma^1_{22}\frac{\partial \tilde{u}}{\partial u} + \Gamma^2_{22}\frac{\partial \tilde{u}}{\partial v} - \tilde{\Gamma}^1_{11}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{u}}{\partial v} - \tilde{\Gamma}^1_{12}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial v} - \tilde{\Gamma}^1_{21}\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial v} - \tilde{\Gamma}^1_{22}\frac{\partial \tilde{v}}{\partial v}\frac{\partial \tilde{v}}{\partial v})\frac{\partial v}{\partial \tilde{u}} \\
\frac{\partial \tilde{s}^1}{\partial \tilde{v}} &= (\frac{\partial s^1}{\partial u}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial \tilde{v}})\frac{\partial \tilde{u}}{\partial u} \\
&+ s^1(\Gamma^1_{11}\frac{\partial \tilde{u}}{\partial u} + \Gamma^2_{11}\frac{\partial \tilde{u}}{\partial v} - \tilde{\Gamma}^1_{11}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{u}}{\partial u} - \tilde{\Gamma}^1_{12}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial u} - \tilde{\Gamma}^1_{21}\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial u} - \tilde{\Gamma}^1_{22}\frac{\partial \tilde{v}}{\partial u}\frac{\partial \tilde{v}}{\partial u})\frac{\partial u}{\partial \tilde{v}}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad s^1(\Gamma^1_{12}\frac{\partial\tilde u}{\partial u} + \Gamma^2_{12}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial u} - \tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u} - \tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u} - \tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial u})\frac{\partial v}{\partial\tilde v}\\
&+ \quad (\frac{\partial s^2}{\partial u}\frac{\partial u}{\partial\tilde v} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial\tilde v})\frac{\partial\tilde u}{\partial v}\\
&+ \quad s^2(\Gamma^1_{21}\frac{\partial\tilde u}{\partial u} + \Gamma^2_{21}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v})\frac{\partial u}{\partial\tilde v}\\
&+ \quad s^2(\Gamma^1_{22}\frac{\partial\tilde u}{\partial u} + \Gamma^2_{22}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial v})\frac{\partial v}{\partial\tilde v}\\
\frac{\partial\tilde s^2}{\partial\tilde u} &= \quad (\frac{\partial s^1}{\partial u}\frac{\partial u}{\partial\tilde u} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial\tilde u})\frac{\partial\tilde v}{\partial u}\\
&+ \quad s^1(\Gamma^1_{11}\frac{\partial\tilde v}{\partial v} + \Gamma^2_{11}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^2_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial u} - \tilde\Gamma^2_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u} - \tilde\Gamma^2_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u} - \tilde\Gamma^2_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial u})\frac{\partial u}{\partial\tilde u}\\
&+ \quad s^1(\Gamma^1_{12}\frac{\partial\tilde v}{\partial u} + \Gamma^2_{12}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial u} - \tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u} - \tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u} - \tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial u})\frac{\partial v}{\partial\tilde u}\\
&+ \quad (\frac{\partial s^2}{\partial u}\frac{\partial u}{\partial\tilde u} + \frac{\partial s^2}{\partial v}\frac{\partial v}{\partial\tilde u})\frac{\partial\tilde v}{\partial v}\\
&+ \quad s^2(\Gamma^1_{21}\frac{\partial\tilde v}{\partial u} + \Gamma^2_{21}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v})\frac{\partial u}{\partial\tilde u}\\
&+ \quad s^2(\Gamma^1_{22}\frac{\partial\tilde v}{\partial u} + \Gamma^2_{22}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial v})\frac{\partial v}{\partial\tilde u}\\
\frac{\partial\tilde s^2}{\partial\tilde v} &= \quad (\frac{\partial s^1}{\partial u}\frac{\partial u}{\partial\tilde v} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial\tilde v})\frac{\partial\tilde v}{\partial u}\\
&+ \quad s^1(\Gamma^1_{11}\frac{\partial\tilde v}{\partial u} + \Gamma^2_{11}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^2_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial u} - \tilde\Gamma^2_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u} - \tilde\Gamma^2_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u} - \tilde\Gamma^2_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial u})\frac{\partial u}{\partial\tilde v}\\
&+ \quad s^1(\Gamma^1_{12}\frac{\partial\tilde v}{\partial u} + \Gamma^2_{12}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial v})\frac{\partial v}{\partial\tilde v}\\
&+ \quad (\frac{\partial s^2}{\partial u}\frac{\partial u}{\partial\tilde v} + \frac{\partial s^1}{\partial v}\frac{\partial v}{\partial\tilde v})\frac{\partial\tilde v}{\partial v}\\
&+ \quad s^2(\Gamma^1_{21}\frac{\partial\tilde v}{\partial u} + \Gamma^2_{21}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v})\frac{\partial u}{\partial\tilde v}\\
&+ \quad s^2(\Gamma^1_{22}\frac{\partial\tilde v}{\partial u} + \Gamma^2_{22}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial v} - \tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v} - \tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial v})\frac{\partial v}{\partial\tilde v}
\end{aligned}
$$

In consequence of

$$
\begin{aligned}
\frac{\partial\tilde u}{\partial u}\frac{\partial u}{\partial\tilde u} &= \frac{\partial\tilde u}{\partial v}\frac{\partial v}{\partial\tilde u} = \frac{\partial\tilde v}{\partial u}\frac{\partial u}{\partial\tilde v} = \frac{\partial\tilde v}{\partial v}\frac{\partial v}{\partial\tilde v} = 1\\
\frac{\partial\tilde u}{\partial u}\frac{\partial u}{\partial\tilde v} &= \frac{\partial\tilde u}{\partial v}\frac{\partial v}{\partial\tilde v} = \frac{\partial\tilde v}{\partial u}\frac{\partial u}{\partial\tilde u} = \frac{\partial\tilde v}{\partial v}\frac{\partial v}{\partial\tilde u} = 0\\
\tilde s^1 &= s^1\frac{\partial\tilde u}{\partial u} + s^2\frac{\partial\tilde u}{\partial v}\\
\tilde s^2 &= s^1\frac{\partial\tilde v}{\partial u} + s^2\frac{\partial\tilde v}{\partial v},
\end{aligned}
$$

we may rearrange the terms as follows

$$
\begin{aligned}
\frac{\partial\tilde s^1}{\partial\tilde u} &= (\frac{\partial s^1}{\partial u} + s^1\Gamma^1_{11} + s^2\Gamma^1_{21})\frac{\partial u}{\partial\tilde u}\frac{\partial\tilde u}{\partial u} + (\frac{\partial s^1}{\partial v} + s^1\Gamma^1_{12} + s^2\Gamma^1_{22})\frac{\partial v}{\partial\tilde u}\frac{\partial\tilde u}{\partial u}\\
&+(\frac{\partial s^2}{\partial u} + s^1\Gamma^2_{11} + s^2\Gamma^2_{21})\frac{\partial u}{\partial\tilde u}\frac{\partial\tilde u}{\partial v} + (\frac{\partial s^2}{\partial v} + s^1\Gamma^2_{12} + s^2\Gamma^2_{22})\frac{\partial v}{\partial\tilde u}\frac{\partial\tilde u}{\partial v}\\
&-s^1\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial u}\frac{\partial u}{\partial\tilde u} - s^1\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u}\frac{\partial u}{\partial\tilde u} - s^1\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial u}\frac{\partial u}{\partial\tilde u} - s^1\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial u}\frac{\partial u}{\partial\tilde u}\\
&-s^1\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial u}\frac{\partial v}{\partial\tilde u} - s^1\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u}\frac{\partial v}{\partial\tilde u} - s^1\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial u}\frac{\partial v}{\partial\tilde u} - s^1\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial u}\frac{\partial v}{\partial\tilde u}\\
&-s^2\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde u}{\partial v}\frac{\partial u}{\partial\tilde u} - s^2\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}\frac{\partial u}{\partial\tilde u} - s^2\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial u}\frac{\partial\tilde v}{\partial v}\frac{\partial u}{\partial\tilde u} - s^2\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial u}\frac{\partial\tilde v}{\partial v}\frac{\partial u}{\partial\tilde u}\\
&-s^2\tilde\Gamma^1_{11}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde u}{\partial v}\frac{\partial v}{\partial\tilde u} - s^2\tilde\Gamma^1_{12}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v}\frac{\partial v}{\partial\tilde u} - s^2\tilde\Gamma^1_{21}\frac{\partial\tilde u}{\partial v}\frac{\partial\tilde v}{\partial v}\frac{\partial v}{\partial\tilde u} - s^2\tilde\Gamma^1_{22}\frac{\partial\tilde v}{\partial v}\frac{\partial\tilde v}{\partial v}\frac{\partial v}{\partial\tilde u}\\
&= (\frac{\partial s^1}{\partial u} + s^1\Gamma^1_{11} + s^2\Gamma^1_{21})\frac{\partial u}{\partial\tilde u}\frac{\partial\tilde u}{\partial u} + (\frac{\partial s^1}{\partial v} + s^1\Gamma^1_{12} + s^2\Gamma^1_{22})\frac{\partial v}{\partial\tilde u}\frac{\partial\tilde u}{\partial u}
\end{aligned}
$$

$$
\begin{aligned}
& +(\frac{\partial s^2}{\partial u}+s^1\Gamma^2_{11}+s^2\Gamma^2_{21})\frac{\partial u}{\partial \tilde u}\frac{\partial \tilde u}{\partial v}+(\frac{\partial s^2}{\partial v}+s^1\Gamma^2_{12}+s^2\Gamma^2_{22})\frac{\partial v}{\partial \tilde u}\frac{\partial \tilde u}{\partial v} \\
& -s^1\tilde\Gamma^1_{11}\frac{\partial \tilde u}{\partial u}-s^2\tilde\Gamma^1_{11}\frac{\partial \tilde u}{\partial v}-s^1\tilde\Gamma^1_{21}\frac{\partial \tilde v}{\partial u}-s^2\tilde\Gamma^1_{21}\frac{\partial \tilde v}{\partial v} \\
= \; & (\frac{\partial s^1}{\partial u}+s^1\Gamma^1_{11}+s^2\Gamma^1_{21})\frac{\partial u}{\partial \tilde u}\frac{\partial \tilde u}{\partial u}+(\frac{\partial s^1}{\partial v}+s^1\Gamma^1_{12}+s^2\Gamma^1_{22})\frac{\partial v}{\partial \tilde u}\frac{\partial \tilde u}{\partial u} \\
& +(\frac{\partial s^2}{\partial u}+s^1\Gamma^2_{11}+s^2\Gamma^2_{21})\frac{\partial u}{\partial \tilde u}\frac{\partial \tilde u}{\partial v}+(\frac{\partial s^2}{\partial v}+s^1\Gamma^2_{12}+s^2\Gamma^2_{22})\frac{\partial v}{\partial \tilde u}\frac{\partial \tilde u}{\partial v} \\
& -\tilde s^1\tilde\Gamma^1_{11}-\tilde s^2\tilde\Gamma^1_{21} \\
\frac{\partial \tilde s^1}{\partial \tilde v} = \; & (\frac{\partial s^1}{\partial u}+s^1\Gamma^1_{11}+s^2\Gamma^1_{21})\frac{\partial u}{\partial \tilde v}\frac{\partial \tilde u}{\partial u}+(\frac{\partial s^1}{\partial v}+s^1\Gamma^1_{12}+s^2\Gamma^1_{22})\frac{\partial v}{\partial \tilde v}\frac{\partial \tilde u}{\partial u} \\
& +(\frac{\partial s^2}{\partial u}+s^1\Gamma^2_{11}+s^2\Gamma^2_{21})\frac{\partial u}{\partial \tilde v}\frac{\partial \tilde u}{\partial v}+(\frac{\partial s^2}{\partial v}+s^1\Gamma^2_{12}+s^2\Gamma^2_{22})\frac{\partial v}{\partial \tilde v}\frac{\partial \tilde u}{\partial v} \\
& -\tilde s^1\tilde\Gamma^1_{12}-\tilde s^2\tilde\Gamma^1_{22} \\
\frac{\partial \tilde s^2}{\partial \tilde u} = \; & (\frac{\partial s^1}{\partial u}+s^1\Gamma^1_{11}+s^2\Gamma^1_{21})\frac{\partial u}{\partial \tilde u}\frac{\partial \tilde v}{\partial u}+(\frac{\partial s^1}{\partial v}+s^1\Gamma^1_{12}+s^2\Gamma^1_{22})\frac{\partial v}{\partial \tilde u}\frac{\partial \tilde v}{\partial u} \\
& +(\frac{\partial s^2}{\partial u}+s^1\Gamma^2_{11}+s^2\Gamma^2_{21})\frac{\partial u}{\partial \tilde u}\frac{\partial \tilde v}{\partial v}+(\frac{\partial s^2}{\partial v}+s^1\Gamma^2_{12}+s^2\Gamma^2_{22})\frac{\partial v}{\partial \tilde u}\frac{\partial \tilde v}{\partial v} \\
& -\tilde s^1\tilde\Gamma^2_{11}-\tilde s^2\tilde\Gamma^2_{21} \\
\frac{\partial \tilde s^2}{\partial \tilde v} = \; & (\frac{\partial s^1}{\partial u}+s^1\Gamma^1_{11}+s^2\Gamma^1_{21})\frac{\partial u}{\partial \tilde v}\frac{\partial \tilde v}{\partial u}+(\frac{\partial s^1}{\partial v}+s^1\Gamma^1_{12}+s^2\Gamma^1_{22})\frac{\partial v}{\partial \tilde v}\frac{\partial \tilde v}{\partial u} \\
& +(\frac{\partial s^2}{\partial u}+s^1\Gamma^2_{11}+s^2\Gamma^2_{21})\frac{\partial u}{\partial \tilde v}\frac{\partial \tilde v}{\partial v}+(\frac{\partial s^2}{\partial v}+s^1\Gamma^2_{12}+s^2\Gamma^2_{22})\frac{\partial v}{\partial \tilde v}\frac{\partial \tilde v}{\partial v} \\
& -\tilde s^1\tilde\Gamma^2_{12}-\tilde s^2\tilde\Gamma^2_{22}
\end{aligned}
$$

and reach an interesting result: if we sum two linear terms to the ordinary derivatives we get quantities that vary linearly under coordinate changes

$$
\begin{aligned}
\frac{\partial \tilde s^1}{\partial \tilde u}+\tilde s^1\tilde\Gamma^1_{11}+\tilde s^2\tilde\Gamma^1_{21} = \; & (\frac{\partial s^1}{\partial u}+s^1\Gamma^1_{11}+s^2\Gamma^1_{21})\frac{\partial u}{\partial \tilde u}\frac{\partial \tilde u}{\partial u}+(\frac{\partial s^1}{\partial v}+s^1\Gamma^1_{12}+s^2\Gamma^1_{22})\frac{\partial v}{\partial \tilde u}\frac{\partial \tilde u}{\partial u} \\
& +(\frac{\partial s^2}{\partial u}+s^1\Gamma^2_{11}+s^2\Gamma^2_{21})\frac{\partial u}{\partial \tilde u}\frac{\partial \tilde u}{\partial v}+(\frac{\partial s^2}{\partial v}+s^1\Gamma^2_{12}+s^2\Gamma^2_{22})\frac{\partial v}{\partial \tilde u}\frac{\partial \tilde u}{\partial v} \\
\frac{\partial \tilde s^1}{\partial \tilde v}+\tilde s^1\tilde\Gamma^1_{12}+\tilde s^2\tilde\Gamma^1_{22} = \; & (\frac{\partial s^1}{\partial u}+s^1\Gamma^1_{11}+s^2\Gamma^1_{21})\frac{\partial u}{\partial \tilde v}\frac{\partial \tilde u}{\partial u}+(\frac{\partial s^1}{\partial v}+s^1\Gamma^1_{12}+s^2\Gamma^1_{22})\frac{\partial v}{\partial \tilde v}\frac{\partial \tilde u}{\partial u} \\
& +(\frac{\partial s^2}{\partial u}+s^1\Gamma^2_{11}+s^2\Gamma^2_{21})\frac{\partial u}{\partial \tilde v}\frac{\partial \tilde u}{\partial v}+(\frac{\partial s^2}{\partial v}+s^1\Gamma^2_{12}+s^2\Gamma^2_{22})\frac{\partial v}{\partial \tilde v}\frac{\partial \tilde u}{\partial v} \\
\frac{\partial \tilde s^2}{\partial \tilde u}+\tilde s^1\tilde\Gamma^2_{11}+\tilde s^2\tilde\Gamma^2_{21} = \; & (\frac{\partial s^1}{\partial u}+s^1\Gamma^1_{11}+s^2\Gamma^1_{21})\frac{\partial u}{\partial \tilde u}\frac{\partial \tilde v}{\partial u}+(\frac{\partial s^1}{\partial v}+s^1\Gamma^1_{12}+s^2\Gamma^1_{22})\frac{\partial v}{\partial \tilde u}\frac{\partial \tilde v}{\partial u} \\
& +(\frac{\partial s^2}{\partial u}+s^1\Gamma^2_{11}+s^2\Gamma^2_{21})\frac{\partial u}{\partial \tilde u}\frac{\partial \tilde v}{\partial v}+(\frac{\partial s^2}{\partial v}+s^1\Gamma^2_{12}+s^2\Gamma^2_{22})\frac{\partial v}{\partial \tilde u}\frac{\partial \tilde v}{\partial v} \\
\frac{\partial \tilde s^2}{\partial \tilde v}+\tilde s^1\tilde\Gamma^2_{12}+\tilde s^2\tilde\Gamma^2_{22} = \; & (\frac{\partial s^1}{\partial u}+s^1\Gamma^1_{11}+s^2\Gamma^1_{21})\frac{\partial u}{\partial \tilde v}\frac{\partial \tilde v}{\partial u}+(\frac{\partial s^1}{\partial v}+s^1\Gamma^1_{12}+s^2\Gamma^1_{22})\frac{\partial v}{\partial \tilde v}\frac{\partial \tilde v}{\partial u} \\
& +(\frac{\partial s^2}{\partial u}+s^1\Gamma^2_{11}+s^2\Gamma^2_{21})\frac{\partial u}{\partial \tilde v}\frac{\partial \tilde v}{\partial v}+(\frac{\partial s^2}{\partial v}+s^1\Gamma^2_{12}+s^2\Gamma^2_{22})\frac{\partial v}{\partial \tilde v}\frac{\partial \tilde v}{\partial v}
\end{aligned} \tag{54}
$$

The quantity of an ordinary derivative along a coordinate plus correction terms is the component of a *covariant derivative*. Often a notation of stroke is used to indicate it, while an ordinary derivative is indicated by a comma

$$
\begin{aligned}
\tilde s^1_{|\tilde u} &= \frac{\partial \tilde s^1}{\partial \tilde u}+\tilde s^1\tilde\Gamma^1_{11}+\tilde s^2\tilde\Gamma^1_{21} = \tilde s^1_{,\tilde u}+\tilde s^1\tilde\Gamma^1_{11}+\tilde s^2\tilde\Gamma^1_{21} \\
\tilde s^1_{|\tilde v} &= \frac{\partial \tilde s^1}{\partial \tilde v}+\tilde s^1\tilde\Gamma^1_{12}+\tilde s^2\tilde\Gamma^1_{22} = \tilde s^1_{,\tilde v}+\tilde s^1\tilde\Gamma^1_{12}+\tilde s^2\tilde\Gamma^1_{22} \\
\tilde s^2_{|\tilde u} &= \frac{\partial \tilde s^2}{\partial \tilde u}+\tilde s^1\tilde\Gamma^2_{11}+\tilde s^2\tilde\Gamma^2_{21} = \tilde s^2_{,\tilde u}+\tilde s^1\tilde\Gamma^2_{11}+\tilde s^2\tilde\Gamma^2_{21} \\
\tilde s^2_{|\tilde v} &= \frac{\partial \tilde s^2}{\partial \tilde v}+\tilde s^1\tilde\Gamma^2_{12}+\tilde s^2\tilde\Gamma^2_{22} = \tilde s^2_{,\tilde v}+\tilde s^1\tilde\Gamma^2_{12}+\tilde s^2\tilde\Gamma^2_{22}
\end{aligned} \tag{55}
$$

By means of Eq. 54 and Eq. 55, we get the result that, differently from the ordinary derivatives (Eq. 53), the covariant derivatives of a contravariant vector are quantified as tensors (Eq. 55)

$$\begin{aligned}
\tilde{s}^1_{|\tilde{u}} &= s^1_{|u}\frac{\partial u}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial u} + s^1_{|v}\frac{\partial v}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial u} + s^2_{|u}\frac{\partial u}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial v} + s^2_{|v}\frac{\partial v}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial u} \\
\tilde{s}^1_{|\tilde{v}} &= s^1_{|u}\frac{\partial u}{\partial \tilde{v}}\frac{\partial \tilde{u}}{\partial u} + s^1_{|v}\frac{\partial v}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial v} + s^2_{|u}\frac{\partial u}{\partial \tilde{v}}\frac{\partial \tilde{u}}{\partial v} + s^2_{|v}\frac{\partial v}{\partial \tilde{v}}\frac{\partial \tilde{u}}{\partial v} \\
\tilde{s}^2_{|\tilde{u}} &= s^1_{|u}\frac{\partial u}{\partial \tilde{u}}\frac{\partial \tilde{v}}{\partial u} + s^1_{|v}\frac{\partial v}{\partial \tilde{u}}\frac{\partial \tilde{v}}{\partial u} + s^2_{|u}\frac{\partial u}{\partial \tilde{u}}\frac{\partial \tilde{v}}{\partial v} + s^2_{|v}\frac{\partial v}{\partial \tilde{u}}\frac{\partial \tilde{v}}{\partial v} \\
\tilde{s}^2_{|\tilde{v}} &= s^1_{|u}\frac{\partial u}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial u} + s^1_{|v}\frac{\partial v}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial u} + s^2_{|u}\frac{\partial u}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial v} + s^2_{|v}\frac{\partial v}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial v}.
\end{aligned}$$

The covariant derivative of a contravariant vector is actually a coordinate-dependent adjustment to ordinary derivative, such that if $\mathbf{s}$ is a contravariant basis vector then

$$\begin{aligned}
\tilde{s}^1_{|\tilde{u}} = \tilde{s}^1_{,\tilde{u}} + \tilde{s}^1\tilde{\Gamma}^1_{11} + \tilde{s}^2\tilde{\Gamma}^1_{21} &= s^1_{|u} = 0 \\
\tilde{s}^1_{|\tilde{v}} = \tilde{s}^1_{,\tilde{v}} + \tilde{s}^1\tilde{\Gamma}^1_{12} + \tilde{s}^2\tilde{\Gamma}^1_{22} &= s^1_{|v} = 0 \\
\tilde{s}^2_{|\tilde{u}} = \tilde{s}^2_{,\tilde{u}} + \tilde{s}^1\tilde{\Gamma}^2_{11} + \tilde{s}^2\tilde{\Gamma}^2_{21} &= s^2_{|u} = 0 \\
\tilde{s}^2_{|\tilde{v}} = \tilde{s}^2_{,\tilde{v}} + \tilde{s}^1\tilde{\Gamma}^2_{12} + \tilde{s}^2\tilde{\Gamma}^2_{22} &= s^2_{|v} = 0
\end{aligned} \tag{56}$$

Knowing a geometrical interpretation for the covariant derivative of a contravariant vector, we may find it by simply taking the orthogonal projections of the ordinary partial derivatives of the contravariant vector $\tilde{\mathbf{s}} = \tilde{s}^1\mathbf{r}_{\tilde{u}} + \tilde{s}^2\mathbf{r}_{\tilde{v}}$ on the tangent plane and applying Eq. 21

$$\begin{aligned}
\tilde{s}^1_{|\tilde{u}} &= \frac{\partial \tilde{\mathbf{s}}}{\partial \tilde{u}}\cdot\mathbf{r}^{\tilde{u}} = \frac{\partial(\tilde{s}^1\mathbf{r}_{\tilde{u}} + \tilde{s}^2\mathbf{r}_{\tilde{v}})}{\partial \tilde{u}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{21}) \\
&= \frac{\partial(\tilde{s}^1\mathbf{r}_{\tilde{u}})}{\partial \tilde{u}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{21}) + \frac{\partial(\tilde{s}^2\mathbf{r}_{\tilde{v}})}{\partial \tilde{u}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{21}) \\
&= (\frac{\partial \tilde{s}^1}{\partial \tilde{u}}\mathbf{r}_{\tilde{u}} + \tilde{s}^1\mathbf{r}_{\tilde{u}\tilde{u}})\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{21}) + (\frac{\partial \tilde{s}^2}{\partial \tilde{u}}\mathbf{r}_{\tilde{v}} + \tilde{s}^2\mathbf{r}_{\tilde{u}\tilde{v}})\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{21}) \\
&= (\frac{\partial \tilde{s}^1}{\partial \tilde{u}}\mathbf{r}_{\tilde{u}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{21}) + \tilde{s}^1\mathbf{r}_{\tilde{u}\tilde{u}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{21})) \\
&\quad +(\frac{\partial \tilde{s}^2}{\partial \tilde{u}}\mathbf{r}_{\tilde{v}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{21}) + \tilde{s}^2\mathbf{r}_{\tilde{u}\tilde{v}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{21})) \\
&= (\frac{\partial \tilde{s}^1}{\partial \tilde{u}}(\tilde{a}_{11}\tilde{a}^{11} + \tilde{a}_{12}\tilde{a}^{21}) + \tilde{s}^1\tilde{\Gamma}^1_{11}) + (\frac{\partial \tilde{s}^2}{\partial \tilde{u}}(\tilde{a}_{12}\tilde{a}^{11} + \tilde{a}_{22}\tilde{a}^{21}) + \tilde{s}^2\tilde{\Gamma}^1_{21}) \\
&= \frac{\partial \tilde{s}^1}{\partial \tilde{u}} + \tilde{s}^1\tilde{\Gamma}^1_{11} + \tilde{s}^2\tilde{\Gamma}^1_{21} = \tilde{s}^1_{,\tilde{u}} + \tilde{s}^1\tilde{\Gamma}^1_{11} + \tilde{s}^2\tilde{\Gamma}^1_{21} \\
\tilde{s}^2_{|\tilde{u}} &= \frac{\partial \tilde{\mathbf{s}}}{\partial \tilde{u}}\cdot\mathbf{r}^{\tilde{v}} = \frac{\partial(\tilde{s}^1\mathbf{r}_{\tilde{u}} + \tilde{s}^2\mathbf{r}_{\tilde{v}})}{\partial \tilde{u}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{21} + \mathbf{r}_{\tilde{v}}\tilde{a}^{22}) = \tilde{s}^2_{,\tilde{u}} + \tilde{s}^1\tilde{\Gamma}^2_{11} + \tilde{s}^2\tilde{\Gamma}^2_{21} \\
\tilde{s}^1_{|\tilde{v}} &= \frac{\partial \tilde{\mathbf{s}}}{\partial \tilde{v}}\cdot\mathbf{r}^{\tilde{u}} = \frac{\partial(\tilde{s}^1\mathbf{r}_{\tilde{u}} + \tilde{s}^2\mathbf{r}_{\tilde{v}})}{\partial \tilde{v}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{11} + \mathbf{r}_{\tilde{v}}\tilde{a}^{12}) = \tilde{s}^1_{,\tilde{v}} + \tilde{s}^1\tilde{\Gamma}^1_{12} + \tilde{s}^2\tilde{\Gamma}^1_{22} \\
\tilde{s}^2_{|\tilde{v}} &= \frac{\partial \tilde{\mathbf{s}}}{\partial \tilde{v}}\cdot\mathbf{r}^{\tilde{v}} = \frac{\partial(\tilde{s}^1\mathbf{r}_{\tilde{u}} + \tilde{s}^2\mathbf{r}_{\tilde{v}})}{\partial \tilde{v}}\cdot(\mathbf{r}_{\tilde{u}}\tilde{a}^{21} + \mathbf{r}_{\tilde{v}}\tilde{a}^{22}) = \tilde{s}^2_{,\tilde{v}} + \tilde{s}^1\tilde{\Gamma}^2_{12} + \tilde{s}^2\tilde{\Gamma}^2_{22}
\end{aligned} \tag{57}$$

Analogously we may obtain the components of the *covariant derivatives of a covariant vector* $\tilde{\mathbf{c}} = \tilde{s}_1\mathbf{r}^{\tilde{u}} + \tilde{s}_2\mathbf{r}^{\tilde{v}}$

$$\begin{aligned}
\tilde{s}_{1|\tilde{u}} &= \frac{\partial \tilde{\mathbf{s}}}{\partial \tilde{u}}\cdot\mathbf{r}_{\tilde{u}} = \frac{\partial(\tilde{s}_1\mathbf{r}^{\tilde{u}} + \tilde{s}_2\mathbf{r}^{\tilde{v}})}{\partial \tilde{u}}\cdot(\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) \\
&= \frac{\partial(\tilde{s}_1\mathbf{r}^{\tilde{u}})}{\partial \tilde{u}}\cdot(\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) + \frac{\partial(\tilde{s}_2\mathbf{r}^{\tilde{v}})}{\partial \tilde{u}}\cdot(\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) \\
&= (\frac{\partial \tilde{s}_1}{\partial \tilde{u}}\mathbf{r}^{\tilde{u}} + \tilde{s}_1\frac{\partial \mathbf{r}^{\tilde{u}}}{\partial \tilde{u}})\cdot(\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) + (\frac{\partial \tilde{s}_2}{\partial \tilde{u}}\mathbf{r}^{\tilde{v}} + \tilde{s}_2\frac{\partial \mathbf{r}^{\tilde{v}}}{\partial \tilde{u}})\cdot(\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) \\
&= (\frac{\partial \tilde{s}_1}{\partial \tilde{u}}\mathbf{r}^{\tilde{u}})\cdot(\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) + \tilde{s}_1\frac{\partial \mathbf{r}^{\tilde{u}}}{\partial \tilde{u}}\cdot(\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) \\
&\quad +(\frac{\partial \tilde{s}_2}{\partial \tilde{u}}\mathbf{r}^{\tilde{v}})\cdot(\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) + \tilde{s}_2\frac{\partial \mathbf{r}^{\tilde{v}}}{\partial \tilde{u}}\cdot(\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12})
\end{aligned} \tag{58}$$

Plugging Eq. 56 into it we get

$$
\begin{aligned}
\tilde{s}_{1|\tilde{u}} &= \frac{\partial \tilde{s}_1}{\partial \tilde{u}} \cdot (\mathbf{r}^{\tilde{u}}\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{u}}\mathbf{r}^{\tilde{v}}\tilde{a}_{12}) + \tilde{s}_1(-\mathbf{r}^{\tilde{u}}\Gamma^1_{11} - \mathbf{r}^{\tilde{v}}\Gamma^1_{21}) \cdot (\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) \\
&\quad + \frac{\partial \tilde{s}_2}{\partial \tilde{u}} \cdot (\mathbf{r}^{\tilde{v}}\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\mathbf{r}^{\tilde{v}}\tilde{a}_{12}) + \tilde{s}_2(-\mathbf{r}^{\tilde{u}}\Gamma^2_{11} - \mathbf{r}^{\tilde{v}}\Gamma^2_{21}) \cdot (\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) \\
&= \frac{\partial \tilde{s}_1}{\partial \tilde{u}} \cdot (\tilde{a}^{11}\tilde{a}_{11} + \tilde{a}^{21}\tilde{a}_{12}) + \tilde{s}_1(-\mathbf{r}^{\tilde{u}}\Gamma^1_{11} \cdot (\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) - \mathbf{r}^{\tilde{v}}\Gamma^1_{21} \cdot (\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12})) \\
&\quad + \frac{\partial \tilde{s}_2}{\partial \tilde{u}} \cdot (\tilde{a}^{12}\tilde{a}_{11} + \tilde{a}^{22}\tilde{a}_{12}) + \tilde{s}_2(-\mathbf{r}^{\tilde{u}}\Gamma^2_{11} \cdot (\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12}) - \mathbf{r}^{\tilde{v}}\Gamma^2_{21} \cdot (\mathbf{r}^{\tilde{u}}\tilde{a}_{11} + \mathbf{r}^{\tilde{v}}\tilde{a}_{12})) \\
&= \frac{\partial \tilde{s}_1}{\partial \tilde{u}} - \tilde{s}_1\Gamma^1_{11} - \tilde{s}_2\Gamma^2_{11} = \tilde{s}_{1,1} - \tilde{s}_1\Gamma^1_{11} - \tilde{s}_2\Gamma^2_{11}
\end{aligned} \tag{59}
$$

The same procedure leads us to the following covariant differentiations

$$
\begin{aligned}
\tilde{s}_{1|\tilde{v}} &= \frac{\partial \tilde{\mathbf{s}}}{\partial \tilde{v}} \cdot \mathbf{r}_{\tilde{u}} = \frac{\partial \tilde{s}_1}{\partial \tilde{v}} - \tilde{s}_1\Gamma^1_{12} - \tilde{s}_2\Gamma^2_{12} = \tilde{s}_{1,2} - \tilde{s}_1\Gamma^1_{12} - \tilde{s}_2\Gamma^2_{12} \\
\tilde{s}_{2|\tilde{u}} &= \frac{\partial \tilde{\mathbf{s}}}{\partial \tilde{u}} \cdot \mathbf{r}_{\tilde{v}} = \frac{\partial \tilde{s}_2}{\partial \tilde{u}} - \tilde{s}_1\Gamma^1_{21} - \tilde{s}_2\Gamma^2_{21} = \tilde{s}_{2,1} - \tilde{s}_1\Gamma^1_{21} - \tilde{s}_2\Gamma^2_{21} \\
\tilde{s}_{2|\tilde{v}} &= \frac{\partial \tilde{\mathbf{s}}}{\partial \tilde{v}} \cdot \mathbf{r}_{\tilde{v}} = \frac{\partial \tilde{s}_2}{\partial \tilde{v}} - \tilde{s}_1\Gamma^1_{22} - \tilde{s}_2\Gamma^2_{22} = \tilde{s}_{2,2} - \tilde{s}_1\Gamma^1_{22} - \tilde{s}_2\Gamma^2_{22}.
\end{aligned} \tag{60}
$$

We have also got these equalities if we differentiated the transformation relations between the covariant vectors $\tilde{\mathbf{c}}$ and $\mathbf{c}$ under the change of basis, as we did in Eq. 53. Moreover, we would have seen that the covariant derivatives of a covariant vector is a covariant tensor of second order

$$
\begin{aligned}
\tilde{s}_{1|\tilde{u}} &= s_{1|u}\frac{\partial u}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{u}} + s_{1|v}\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{u}} + s_{2|u}\frac{\partial v}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{u}} + s_{2|v}\frac{\partial v}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{u}} \\
\tilde{s}_{1|\tilde{v}} &= s_{1|u}\frac{\partial u}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{v}} + s_{1|v}\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} + s_{2|u}\frac{\partial v}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{v}} + s_{2|v}\frac{\partial v}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} \\
\tilde{s}_{2|\tilde{u}} &= s_{1|u}\frac{\partial u}{\partial \tilde{v}}\frac{\partial u}{\partial \tilde{u}} + s_{1|v}\frac{\partial u}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{u}} + s_{2|u}\frac{\partial v}{\partial \tilde{v}}\frac{\partial u}{\partial \tilde{u}} + s_{2|v}\frac{\partial v}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{u}} \\
\tilde{s}_{2|\tilde{v}} &= s_{1|u}\frac{\partial u}{\partial \tilde{v}}\frac{\partial u}{\partial \tilde{v}} + s_{1|v}\frac{\partial u}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{v}} + s_{2|u}\frac{\partial v}{\partial \tilde{v}}\frac{\partial u}{\partial \tilde{v}} + s_{2|v}\frac{\partial v}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{v}}.
\end{aligned}
$$

Applying Eq. 60 we may rewrite Eq. 28 in the form

$$
\begin{aligned}
\mathbf{r}_{uu} - \Gamma^1_{11}\mathbf{r}_u - \Gamma^2_{11}\mathbf{r}_v = b_{11}\mathbf{n} &\implies \mathbf{r}_{u|u} = b_{11}\mathbf{n} \\
\mathbf{r}_{uv} - \Gamma^1_{12}\mathbf{r}_u - \Gamma^2_{12}\mathbf{r}_v = b_{12}\mathbf{n} &\implies \mathbf{r}_{u|v} = b_{12}\mathbf{n} \\
\mathbf{r}_{vu} - \Gamma^1_{21}\mathbf{r}_u - \Gamma^2_{21}\mathbf{r}_v = b_{21}\mathbf{n} &\implies \mathbf{r}_{v|u} = b_{21}\mathbf{n} \\
\mathbf{r}_{vv} - \Gamma^1_{22}\mathbf{r}_v - \Gamma^2_{22}\mathbf{r}_v = b_{22}\mathbf{n} &\implies \mathbf{r}_{v|v} = b_{22}\mathbf{n}.
\end{aligned} \tag{61}
$$

The Christoffel symbols of the second kind are defined in terms of the components of the metric tensor and their derivatives .

Eq. 59 and Eq. 60 show us that for a covariant vector its covariant derivatives of higher order differ from its ordinary ones only by subtracting the linear terms of the Christoffel symbols of the second kind. Extending to the partial derivatives of the covariant tensors of second order, we have [30]

$$
\begin{aligned}
\theta_{\alpha\beta|u} &= \frac{\partial \theta_{\alpha\beta}}{\partial u} - \theta_{1\beta}\Gamma^1_{\alpha 1} - \theta_{\alpha 1}\Gamma^1_{\beta 1} - \theta_{2\beta}\Gamma^2_{\alpha 1} - \theta_{\alpha 2}\Gamma^2_{\beta 1} \\
\theta_{\alpha\beta|v} &= \frac{\partial \theta_{\alpha\beta}}{\partial v} - \theta_{1\beta}\Gamma^1_{\alpha 2} - \theta_{\alpha 1}\Gamma^1_{\beta 2} - \theta_{2\beta}\Gamma^2_{\alpha 2} - \theta_{\alpha 2}\Gamma^2_{\beta 2}.
\end{aligned} \tag{62}
$$

Denoting the *covariant differential* of $\theta_{\alpha\beta}$ for any displacement $d\mathbf{e} = du\mathbf{r}_u + dv\mathbf{r}_v$ by $D\theta_{\alpha\beta}$, we have

$$
\begin{aligned}
D\theta_{\alpha\beta} &= \theta_{\alpha\beta|u}du + \theta_{\alpha\beta|v}dv \\
&= (\frac{\partial \theta_{\alpha\beta}}{\partial u} - \theta_{1\beta}\Gamma^1_{\alpha 1} - \theta_{\alpha 1}\Gamma^1_{\beta 1} - \theta_{2\beta}\Gamma^2_{\alpha 1} - \theta_{\alpha 2}\Gamma^2_{\beta 1})du + \\
&\quad (\frac{\partial \theta_{\alpha\beta}}{\partial v} - \theta_{1\beta}\Gamma^1_{\alpha 2} - \theta_{\alpha 1}\Gamma^1_{\beta 2} - \theta_{2\beta}\Gamma^2_{\alpha 2} - \theta_{\alpha 2}\Gamma^2_{\beta 2})dv \\
&= (\frac{\partial \theta_{\alpha\beta}}{\partial u}du + \frac{\partial \theta_{\alpha\beta}}{\partial v}dv)
\end{aligned}
$$

$$
\begin{aligned}
& -(\theta_{1\beta}\Gamma^1_{\alpha 1} + \theta_{\alpha 1}\Gamma^1_{\beta 1} + \theta_{2\beta}\Gamma^2_{\alpha 1} + \theta_{\alpha 2}\Gamma^2_{\beta 1})du \\
& -(\theta_{1\beta}\Gamma^1_{\alpha 2} + \theta_{\alpha 1}\Gamma^1_{\beta 2} + \theta_{2\beta}\Gamma^2_{\alpha 2} + \theta_{\alpha 2}\Gamma^2_{\beta 2})dv \\
= \; & d\theta_{\alpha\beta} - (\theta_{1\beta}\Gamma^1_{\alpha 1} + \theta_{\alpha 1}\Gamma^1_{\beta 1} + \theta_{2\beta}\Gamma^2_{\alpha 1} + \theta_{\alpha 2}\Gamma^2_{\beta 1})du \\
& -(\theta_{1\beta}\Gamma^1_{\alpha 2} + \theta_{\alpha 1}\Gamma^1_{\beta 2} + \theta_{2\beta}\Gamma^2_{\alpha 2} + \theta_{\alpha 2}\Gamma^2_{\beta 2})dv \qquad (63)
\end{aligned}
$$

Similarly, for a contravariant tensor of second order, $\tau^{\alpha\beta}$, we have

$$
\begin{aligned}
\tau^{\alpha\beta}_{|u} &= \frac{\partial \tau^{\alpha\beta}}{\partial u} + \tau^{1\beta}\Gamma^{\alpha}_{11} + \tau^{\alpha 1}\Gamma^{\beta}_{11} + \tau^{2\beta}\Gamma^{\alpha}_{21} + \tau^{\alpha 2}\Gamma^{\beta}_{21} \\
\tau^{\alpha\beta}_{|v} &= \frac{\partial \tau^{\alpha\beta}}{\partial v} + \tau^{1\beta}\Gamma^{\alpha}_{11} + \tau^{\alpha 1}\Gamma^{\beta}_{11} + \tau^{2\beta}\Gamma^{\alpha}_{21} + \tau^{\alpha 2}\Gamma^{\beta}_{21}. \qquad (64)
\end{aligned}
$$

and the following covariant differential

$$
\begin{aligned}
D\tau^{\alpha\beta} &= \tau^{\alpha\beta}_{|u}du + \tau^{\alpha\beta}_{|v}dv \\
&= (\frac{\partial \tau^{\alpha\beta}}{\partial u} + \tau^{1\beta}\Gamma^{\alpha}_{11} + \tau^{\alpha 1}\Gamma^{\beta}_{11} + \tau^{2\beta}\Gamma^{\alpha}_{21} + \tau^{\alpha 2}\Gamma^{\beta}_{21})du + \\
&\quad (\frac{\partial \tau^{\alpha\beta}}{\partial v} + \tau^{1\beta}\Gamma^{\alpha}_{11} + \tau^{\alpha 1}\Gamma^{\beta}_{11} + \tau^{2\beta}\Gamma^{\alpha}_{21} + \tau^{\alpha 2}\Gamma^{\beta}_{21})dv \\
&= (\frac{\partial \tau^{\alpha\beta}}{\partial u}du + \frac{\partial \tau^{\alpha\beta}}{\partial v}dv) \\
&\quad +(\tau^{1\beta}\Gamma^{\alpha}_{11} + \tau^{\alpha 1}\Gamma^{\beta}_{11} + \tau^{2\beta}\Gamma^{\alpha}_{21} + \tau^{\alpha 2}\Gamma^{\beta}_{21})du \\
&\quad +(\tau^{1\beta}\Gamma^{\alpha}_{11} + \tau^{\alpha 1}\Gamma^{\beta}_{11} + \tau^{2\beta}\Gamma^{\alpha}_{21} + \tau^{\alpha 2}\Gamma^{\beta}_{21})dv \\
&= d\tau^{\alpha\beta} + (\tau^{1\beta}\Gamma^{\alpha}_{11} + \tau^{\alpha 1}\Gamma^{\beta}_{11} + \tau^{2\beta}\Gamma^{\alpha}_{21} + \tau^{\alpha 2}\Gamma^{\beta}_{21})du \\
&\quad +(\tau^{1\beta}\Gamma^{\alpha}_{11} + \tau^{\alpha 1}\Gamma^{\beta}_{11} + \tau^{2\beta}\Gamma^{\alpha}_{21} + \tau^{\alpha 2}\Gamma^{\beta}_{21})dv, \qquad (65)
\end{aligned}
$$

For a general tensor of a type (r, s) that is defined in an n-dimensional space, in which the coordinates of a pont $\mathcal{P}$ are denoted by $(u^1, u^2, \cdots, u^n)$, its (partial) covariant derivative is defined to be

$$
\tau^{j_1\cdots j_r}_{l_1\cdots l_s|k} = \frac{\partial \tau^{j_1\cdots j_r}_{l_1\cdots l_s}}{\partial u^k} + \Gamma^{j_\alpha}_{mk}\tau^{j_1\cdots j_{\alpha-1}mj_{\alpha+1}\cdots j_r}_{l_1\cdots l_s} - \Gamma^{m}_{l_\beta k}\tau^{j_1\cdots j_r}_{l_1\cdots l_{\alpha-1}ml_{\alpha+1}\cdots l_s} \qquad (66)
$$

To conclude this section, it is worth mentioning some basic laws of covariant derivatives [30]

Closure rule: The covariant differentiation of a tensor field of a type (r, s) is a tensor a type$(r, s+1)$; in particular, the covariant differentiation of a scalar field is its ordinary differential.

Sum rule: The covariant differentiation of the sum of two tensor fields of the same type is the sum of the covariant differentiations of these fields.

Product rule: The covariant differentiation of the product of any two tensor fields is given in terms of the covariant differentiations of these fields by a rule which is formally identical with the product rule of partial/ordinary differentials.

The covariant derivative, indeed, is a way of specifying a derivative along a (unit) tangent vector $\mathbf{v}$ of the surface $\mathcal{S}$. It is namely a *directional derivative*, or the rate at which a function changes at a point $\mathcal{P}$ in the direction $\mathbf{v}$. Many of the familiar properties of the ordinary derivative hold for this class of derivatives. These include, for any differentiable functions f and g defined in the neighborhood of $\mathcal{P}$:

Sum rule: $(f + g)_{|v} = f_{|v} + g_{|v}$,

Constant factor rule: For any constant c, $(cf)_{|v} = cf_{|v}$, and

Product rule: $(fg)_{|v} = gf_{|v} + fg_{|v}$.

Finally, we should mention an equality that is useful to this work. If the direction $\mathbf{d}$ is expressed in terms of the basis vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ on the tangent plane to $\mathcal{S}$ at $\mathcal{P}$, that is $\mathbf{d} = \lambda\mathbf{r}_u + \mu\mathbf{r}_v$, we have the following directional derivation rule

$$
D_{\mathbf{d}}f = D_{\lambda\mathbf{r}_u+\mu\mathbf{r}_v}f = \lambda f_u + \mu f_v.
$$

2.4.5 Derivatives of Metric and Curvature Tensors

Plugging the elements of the metric tensor in Eq. 62, we obtain from the equalities in Eq. 34 as zero tensors their covariant derivatives [29]

$$\begin{aligned} a_{\alpha\beta|u} &= \frac{\partial a_{\alpha\beta}}{\partial u} - \Gamma_{\alpha 1\beta} - \Gamma_{\beta 1\alpha} = 0 \\ a_{\alpha\beta|v} &= \frac{\partial a_{\alpha\beta}}{\partial v} - \Gamma_{\alpha 2\beta} - \Gamma_{\beta 2\alpha} = 0. \end{aligned} \tag{67}$$

If we plug the elements of the curvature tensor in Eq. 62 and apply Eq. 33 and Eq. 48, we have [22]

$$\begin{aligned} b_{\alpha\beta|u} &= \frac{\partial b_{\alpha\beta}}{\partial u} - b_{1\beta}\Gamma^1_{\alpha 1} - b_{\alpha 1}\Gamma^1_{\beta 1} - b_{2\beta}\Gamma^2_{\alpha 1} - b_{\alpha 2}\Gamma^2_{\beta 1} \\ &= \frac{\partial b_{\alpha\beta}}{\partial u} - b_{1\beta}(a^{11}\Gamma_{\alpha 11} + a^{21}\Gamma_{\alpha 12}) - b_{\alpha 1}(a^{11}\Gamma_{\beta 11} + a^{21}\Gamma_{\beta 12}) - \\ &\quad b_{2\beta}(a^{12}\Gamma_{\alpha 11} + a^{22}\Gamma_{\alpha 12}) - b_{\alpha 2}(a^{12}\Gamma_{\beta 11} + a^{22}\Gamma_{\beta 12}) \\ &= \frac{\partial(\mathbf{r}_{w^\alpha w^\beta}\cdot\mathbf{n})}{\partial u} - \Gamma_{\alpha 11}b^1_\beta - \Gamma_{\alpha 12}b^2_\beta - \Gamma_{\beta 11}b^1_\alpha - \Gamma_{\beta 12}b^2_\alpha \\ &= \mathbf{r}_{w^\alpha w^\beta u}\cdot\mathbf{n} + \mathbf{r}_{w^\alpha w^\beta}\cdot\frac{\partial\mathbf{n}}{\partial u} - \mathbf{r}_{w^\alpha u}\cdot(\mathbf{r}_u b^1_\beta + \mathbf{r}_v b^2_\beta) - \mathbf{r}_{w^\beta u}\cdot(\mathbf{r}_u b^1_\alpha + \mathbf{r}_v b^2_\alpha) \\ &= \mathbf{r}_{w^\alpha w^\beta u}\cdot\mathbf{n} + \mathbf{r}_{w^\alpha w^\beta}\cdot\frac{\partial\mathbf{n}}{\partial u} - \mathbf{r}_{w^\alpha u}\cdot(\mathbf{r}_u b^1_\beta + \mathbf{r}_v b^2_\beta) - \mathbf{r}_{w^\beta u}\cdot(\mathbf{r}_u b^1_\alpha + \mathbf{r}_v b^2_\alpha) \\ &= \mathbf{r}_{w^\alpha w^\beta u}\cdot\mathbf{n} + \mathbf{r}_{w^\alpha w^\beta}\cdot\frac{\partial\mathbf{n}}{\partial u} + \mathbf{r}_{w^\alpha u}\cdot\frac{\partial\mathbf{n}}{\partial w^\beta} + \mathbf{r}_{w^\beta u}\cdot\frac{\partial\mathbf{n}}{\partial w^\alpha} \\ b_{\alpha\beta|v} &= \frac{\partial b_{\alpha\beta}}{\partial v} - b_{1\beta}\Gamma^1_{\alpha 2} - b_{\alpha 1}\Gamma^1_{\beta 2} - b_{2\beta}\Gamma^2_{\alpha 2} - b_{\alpha 2}\Gamma^2_{\beta 2} \\ &= \frac{\partial b_{\alpha\beta}}{\partial v} - b_{1\beta}(a^{11}\Gamma_{\alpha 21} + a^{21}\Gamma_{\alpha 22}) - b_{\alpha 1}(a^{11}\Gamma_{\beta 21} + a^{21}\Gamma_{\beta 22}) - \\ &\quad b_{2\beta}(a^{12}\Gamma_{\alpha 21} + a^{22}\Gamma_{\alpha 22}) - b_{\alpha 2}(a^{12}\Gamma_{\beta 21} + a^{22}\Gamma_{\beta 22}) \\ &= \frac{\partial(\mathbf{r}_{w^\alpha w^\beta}\cdot\mathbf{n})}{\partial v} - \Gamma_{\alpha 21}b^1_\beta - \Gamma_{\alpha 22}b^2_\beta - \Gamma_{\beta 21}b^1_\alpha - \Gamma_{\beta 22}b^2_\alpha \\ &= \mathbf{r}_{w^\alpha w^\beta v}\cdot\mathbf{n} + \mathbf{r}_{w^\alpha w^\beta}\cdot\frac{\partial\mathbf{n}}{\partial v} - \mathbf{r}_{w^\alpha v}\cdot(\mathbf{r}_u b^1_\beta + \mathbf{r}_v b^2_\beta) - \mathbf{r}_{w^\beta v}\cdot(\mathbf{r}_u b^1_\alpha + \mathbf{r}_v b^2_\alpha) \\ &= \mathbf{r}_{w^\alpha w^\beta v}\cdot\mathbf{n} + \mathbf{r}_{w^\alpha w^\beta}\cdot\frac{\partial\mathbf{n}}{\partial v} - \mathbf{r}_{w^\alpha v}\cdot(\mathbf{r}_u b^1_\beta + \mathbf{r}_v b^2_\beta) - \mathbf{r}_{w^\beta v}\cdot(\mathbf{r}_u b^1_\alpha + \mathbf{r}_v b^2_\alpha) \\ &= \mathbf{r}_{w^\alpha w^\beta v}\cdot\mathbf{n} + \mathbf{r}_{w^\alpha w^\beta}\cdot\frac{\partial\mathbf{n}}{\partial v} + \mathbf{r}_{w^\alpha v}\cdot\frac{\partial\mathbf{n}}{\partial w^\beta} + \mathbf{r}_{w^\beta v}\cdot\frac{\partial\mathbf{n}}{\partial w^\alpha} \end{aligned}$$

where $u = w^1$ and $v = w^2$. This leads us to the following equalities

$$\begin{aligned} b_{11|u} &= \mathbf{r}_{uuu}\cdot\mathbf{n} + 3\mathbf{r}_{uu}\cdot\mathbf{n}_u \\ b_{11|v} = b_{12|u} = b_{21|u} &= \mathbf{r}_{uuv}\cdot\mathbf{n} + \mathbf{r}_{uu}\cdot\mathbf{n}_v + 2\mathbf{r}_{uv}\cdot\mathbf{n}_u \\ b_{22|u} = b_{12|v} = b_{21|v} &= \mathbf{r}_{uvv}\cdot\mathbf{n} + \mathbf{r}_{vv}\cdot\mathbf{n}_u + 2\mathbf{r}_{uv}\cdot\mathbf{n}_v \\ b_{22|v} &= \mathbf{r}_{vvv}\cdot\mathbf{n} + 3\mathbf{r}_{vv}\cdot\mathbf{n}_v. \end{aligned} \tag{68}$$

2.4.6 Gaussian and Mean Curvatures

According to the Euler's theorem, the normal curvature k_n for any direction can be given in terms of the principal curvatures

$$k_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta, \tag{69}$$

where θ is the angle between a direction of interest and the principal direction corresponding to κ_1. The two theorems, Mousnier's and Euler's, give full information concerning the curvature of any curve through $\mathcal{P}$ on the surface $\mathcal{S}$.

The product

$$K = \kappa_1\kappa_2 = \frac{b_{11}b_{22} - b_{12}^2}{(a_{11}a_{22} - (a_{12})^2)}. \tag{70}$$

of the two principal normal curvatures is called *Gaussian curvature* and their arithmetic mean

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{a_{11}b_{22} - 2a_{12}b_{12} + a_{22}b_{11}}{2(a_{11}a_{22} - (a_{12})^2)} \tag{71}$$

is called the *mean curvature.*

Conversely,

$$\begin{aligned} k_1 &= H - \sqrt{H^2 - K} \\ k_2 &= H + \sqrt{H^2 - K}. \end{aligned} \tag{72}$$

In some sense, Gaussian curvature measures how far a surface is from being Euclidean plane, since it relates the small radius ϵ around the point $\mathcal{P}$ and their circumference $L(\mathcal{C})$ (Figure 6.a). Another geometrical interpretation may be derived from the Gauss map $N\colon \mathcal{S} \to \mathcal{S}^2$, which expresses the unit normal vector at each point $\mathcal{P}$ on the surface $\mathcal{S}$ in terms of a vector position of a unit sphere $\mathcal{S}^2 \in \Re^3$. If the Gaussian curvature at $\mathcal{P}$ does not vanish and Gaussian curvature of its connected neighborhood does not change sign,

$$K = lim_{\mathcal{A}\to 0}\frac{\mathcal{A}'}{\mathcal{A}},$$

where $\mathcal{A}$ is the area in $\mathcal{S}$ containig $\mathcal{P}$ and $\mathcal{A}'$ is the area of its image in $\mathcal{S}^2$ (Figure 6.a). The third geometrical interpretation is due to Gauss and O. Bonnet

Theorem 2.1 (Gauss-Bonnet) *If Gaussian curvature K of a surface is continuous in a simply connected region $\mathcal{R}$ bounded by a closed curve $\mathcal{C}$ composed of k smooth arcs intersecting at exterior angles β_1, β_2,β_3, $\cdots$, β_k, then*

$$\int_{\mathcal{C}} k_g ds + \int\int_{\mathcal{R}} K dA = 2\pi - \sum_{i=1}^{k} \beta_i$$

where k_g represents the geodesic curvature of the arcs.

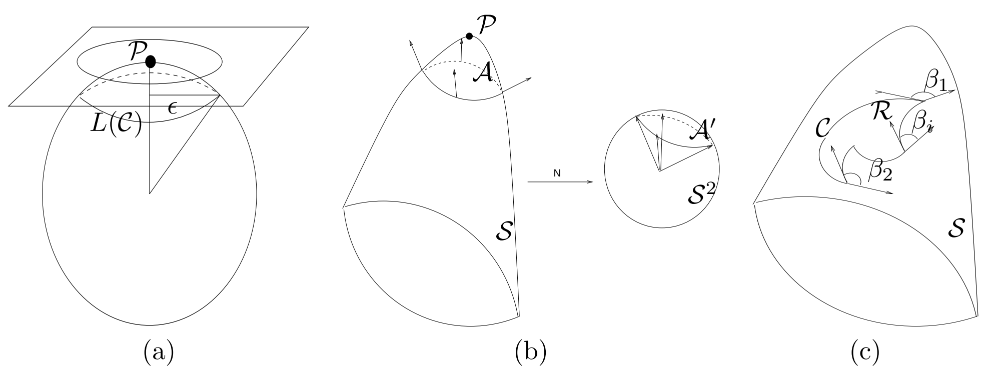

Figure 6: Geometrical interpretation of Gaussian curvature: (a) distance to Euclidean plane; (b) limit of the area ratio; (c) sum of the exterior angles.

While Gaussian curvature is related with the ratio between the coverage of the normal vectors (the image of the Gauss map) and the area defined by the corresponding points (the domain of the Gauss map), the mean curvature may be associated to the variation of the area $\mathcal{A}$ bounded by a closed curve $\mathcal{C}(u(t), v(t))$ on $\mathcal{S}$, with respect to the distance $h(u(t), v(t))$ on the direction $\mathbf{n}(u(t), v(t))$ (Figure 7)

$$\frac{d(\mathcal{A})}{dh} = -2\int_{\mathcal{C}} h(u(t), v(t))H(u(t), v(t))dt \tag{73}$$

The sign of the normal curvature is dependent on the orientation of the normal vector $\mathbf{n}$, and thus on the orientation of the surface $\mathcal{S}$. It is positive if the curve normal vector and the normal vector of $\mathcal{S}$ lie on the same side of the curve ; otherwise it is negative. Therefore, the sign of the mean curvature H depends on the surface orientation, and therefore on its parametrization.

On the basis of Gaussain and the mean curvatures, we may classify a point of a surface $\mathcal{S}$ in

elliptic : if K, $H > 0$;

hyperbolic : if K, $H < 0$;

parabolic : if $K = 0$ and $H \neq 0$; and

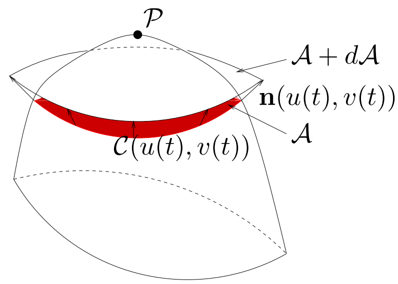

Figure 7: Geometrical interpretation of the mean curvature.

planar : if $K = H = 0$.

In addition, we say that a point is *umbilical* if $\kappa_1 = \kappa_2$.

Eq. 28 and Eq. 50 are not independent. They must satisfy the compatibility conditions

$$\mathbf{r}_{uuv} = \mathbf{r}_{uvu} \quad \mathbf{r}_{uvv} = \mathbf{r}_{vvu} \quad \mathbf{n}_{uv} = \mathbf{n}_{vu},$$

from which we may derive the Mainardi-Codazzi equations

$$\begin{aligned} \frac{\partial b_{11}}{\partial v} - \frac{\partial b_{12}}{\partial u} &= b_{11}\Gamma^1_{12} + b_{12}(\Gamma^2_{12} - \Gamma^1_{11}) - b_{22}\Gamma^2_{11} \\ \frac{\partial b_{12}}{\partial v} - \frac{\partial b_{22}}{\partial u} &= b_{11}\Gamma^1_{22} + b_{12}(\Gamma^2_{22} - \Gamma^1_{12}) - b_{22}\Gamma^2_{12} \end{aligned} \tag{74}$$

A natural question is whether the converse also holds, that is whether the knowledge of the first and second fundamental form determines a surface locally. The answer of this question is due to O. Bonnet.

Theorem 2.2 (Fundamental Theorem) *If a_{11}, a_{12}, a_{22} and b_{11}, b_{12}, b_{22} are given as functions of u and v, sufficiently differentiable, which satisfy the Mainardi-Codazzi equations, while $a_{11}a_{22} - a_{12}^2 \neq 0$, then there exists a surface which admits as its first and second fundamental forms $I = a_{11}du^2 + 2a_{12}dudv + a_{22}dv^2$ and $II = b_{11}du^2 + 2b_{12}dudv + b_{22}dv^2$, respectively. This surface is uniquely determined except for its position is space.*

2.4.7 Einstein Notation

To simplify the expressions presented in the previous sections, it is convenient to adopt the Einstein notation or Einstein summation convention, which was introduced by Albert Einstein in 1916 in his development of the Theory of Relativity.

Contravariant components : Upper indices a^i are used to label them.

Covariant components : Lower indices a_i are used to label them.

Free Indices: Instead of writing all possible values for an index, the strategy is to define a free index j that may take on any particular value

$$y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \equiv y_i = a_{ij}x_j$$

Einstein Summation Convention: Any expression involving a twice-repeated index shall automatically stand for its sum over all of its possible values.

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum_{i=1}^{n} a_ix_i \equiv y = a_ix_i$$

If in particular we choose as the coordinates (u^1, u^2) and adopt Greek indices with the range 1, 2 for denoting the components of metric and curvature tensors, then several representations given before may be largely simplified. Let us rewrite some formulae from Section 2.4 in Einstein notation:

1. Eq. 9 assumes the following aspect

$$\begin{aligned} I(\alpha(t)) = d\alpha(t)\cdot d\alpha(t) &= a_{11}\frac{du^1}{dt}\frac{du^1}{dt} + a_{12}\frac{du^1}{dt}\frac{du^2}{dt} + a_{21}\frac{du^1}{dt}\frac{du^2}{dt} + a_{22}\frac{du^2}{dt}\frac{du^2}{dt} \\ &= \sum_{\beta=1}^{2}(\sum_{\alpha=1}^{2} a_{\alpha\beta}\frac{du^\alpha}{dt}\frac{du^\beta}{dt}) \\ \equiv d\alpha(t)\cdot d\alpha(t) &= a_{\alpha\beta}\frac{du^\alpha}{dt}\frac{du^\beta}{dt}; \end{aligned} \tag{75}$$

2. Eq. 45 becomes

$$II(d^2\alpha(t)\cdot \mathbf{n}) = b_{\alpha\beta}\frac{du^\alpha}{dt}\frac{du^\beta}{dt}; \tag{76}$$

3. Eq. 30 may be reduced to a streamline

$$\Gamma^\gamma_{\alpha\beta} = \mathbf{r}_{\alpha\beta}\cdot \mathbf{r}_\lambda a^{\lambda\gamma} = \mathbf{r}_{\alpha\beta}\cdot \mathbf{r}^\gamma \tag{77}$$

and the equalities in Eq. 31 (Christoffel symbols of the first kind) compacted in the form

$$\Gamma_{\alpha\beta\gamma} = \mathbf{r}_{\alpha\beta}\cdot \mathbf{r}_\gamma \tag{78}$$

4. Eq. 61, Eq. 39, and Eq. 48 may be compacted to

$$\mathbf{r}_{\alpha,\beta} = \Gamma^\rho_{\alpha\beta}\mathbf{r}_\rho + b_{\alpha\beta}\mathbf{n} \quad\equiv\quad \mathbf{r}_{\alpha|\beta} = b_{\alpha\beta}\mathbf{n} \tag{79}$$

$$\mathbf{r}^\alpha_{,\beta} = -\Gamma^\alpha_{\rho\beta}\mathbf{r}^\rho + b^\alpha_\beta\mathbf{n} \quad\equiv\quad \mathbf{r}^\alpha_{|\beta} = b^\alpha_\beta\mathbf{n} \tag{80}$$

$$\mathbf{n}_{,\beta} = -b^\alpha_\beta\mathbf{r}_\alpha; \tag{81}$$

5. The components of the derivative covariant of a contravariant vector **s** (Eq 55) takes the form

$$\tilde{s}^\alpha_{|\beta} = \tilde{s}^\alpha_{,\beta} + \tilde{s}^\rho\tilde{\Gamma}^\alpha_{\rho\beta}; \tag{82}$$

while the components of the derivative covariant of a covariant vector **s** (Eq. 60) assumes the form

$$\tilde{s}_{\alpha|\beta} = \tilde{s}_{\alpha,\beta} - \tilde{s}_\rho\tilde{\Gamma}^\rho_{\alpha\beta}. \tag{83}$$

6. The coefficients of Eq. 50 can be expressed as

$$b^\beta_\alpha = a^{\gamma\beta}b_{\alpha\gamma} \tag{84}$$

2.5 Discrete Differential Geometry

As we are working on surface samples, more precisely on meshes of arbitrary topology, it is necessary to replace the geometric quantity relations available for its continuous counterpart with equivalent relations on discrete domain. In this section we show how to estimate from the discrete data samples the geometric quantities of interest on the basis of ideas presented in [27, 39]. In [3] detailed comparisons with other approaches are provided.

2.5.1 Normal Vector

A variety of algorithms for estimating vertex normal vectors on a sampled 2D geometry embedded in $\Re^3$ is given in [27]. The eldest one, was proposed by Henri Gouraud [21]. Considering that the normal vector of each of its adjacent faces contributes equally to the normal vector of a vertex v, one may simply summing up the normal vectors $\mathbf{N}_i$ of all n faces incident to v and normalizing the result

$$\mathbf{n}_v = \frac{\sum_{i=1}^n \mathbf{N}_i}{|\sum_{i=1}^n \mathbf{N}_i|} \tag{85}$$

However, Max showed in [31] that taking the area A_i of the face i divided by the squared lengths of its two edges, e_i and $e_{((i+1)\ mod\ (n+1))+1}$, as the weight of the face normal $\mathbf{N}_i$ accumulated to the vertex normal $\mathbf{n}_v$, produces more accurate estimates (Figure 8). For vertices that lie on a sphere this weighting procedure is exact. Hence, we applied this weighting procedure as [39] in our implementation

$$\mathbf{n}_v = \sum_{i=1}^n \mathbf{N}_i \frac{A_i}{\sqrt{(e_i\cdot e_i)(e_{((i+1)\ mod\ (n+1))+1}\cdot e_{((i+1)\ mod\ (n+1))+1})}} \tag{86}$$

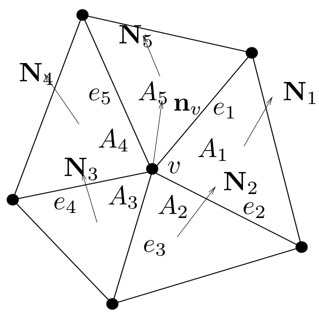

Figure 8: Normal vector of a vertex as the weighted sum of normal vectors of its adjacent faces.

2.5.2 Coefficients of a Curvature Tensor

Providing approximated normal vector at all vertices, we estimate their variation along each edge $\mathbf{e}_i = v_i - v$ incident to a vertex v by (Figure 9)

$$\Delta\mathbf{n}_i = \mathbf{n}_i - \mathbf{n}.$$

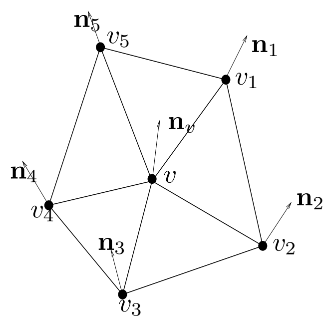

Figure 9: 1-ring vicinity of a vertex.

Differently from the two-pass procedure proposed by Rusinkiewicz [39], in which per-vertex curvature is obtained from the "Voronoi area" weighted sum of per-face curvatures, we devised a one-pass algorithm. We consider that each vertex has its own orthonormal coordinate system defined as follows

$$\begin{aligned} \mathbf{t} &= \frac{v_0 v}{|v_0 v|} \\ \mathbf{b} &= \mathbf{n} \times \mathbf{t} \\ \mathbf{n} &= \mathbf{n}. \end{aligned} \tag{87}$$

Using the triple (t, b, n) as the coordinates of this reference system with origin at v and considering that $\mathbf{e}_i$ is a tangent vector to a curve on the underlying surface, we may assume that

$$\begin{aligned} \mathbf{e}_i &\approx \mathbf{t}\Delta t + \mathbf{b}\Delta b \\ &= \mathbf{t} t_i + \mathbf{b} b_i \end{aligned} \tag{88}$$

where t_i and b_i can be obtained from the "orthogonal projections" of $\mathbf{e}_i$ on the reference vectors $\mathbf{t}$ and $\mathbf{b}$, respectively,

$$\begin{aligned} t_i &= \mathbf{e}_i \cdot \mathbf{t} \\ b_i &= \mathbf{e}_i \cdot .\mathbf{b}. \end{aligned}$$

Observe that Eq. 88 is indeed a discrete version of Eq. 8.

Moreover, if we discretize Eq. 48 as follows

$$\begin{aligned} d_{\mathbf{t}}\mathbf{n} &\approx -b_1^1\mathbf{t} - b_2^1\mathbf{b} \\ d_{\mathbf{b}}\mathbf{n} &\approx -b_1^2\mathbf{t} - b_2^2\mathbf{b}, \end{aligned} \tag{89}$$

we also have

$$d\mathbf{n} = (-b_1^1\mathbf{t} - b_2^1\mathbf{b})t_i + (-b_1^2\mathbf{t} - b_2^2\mathbf{b})b_i. \tag{90}$$

Once, per construction, the basis vectors $\mathbf{t}$ and $\mathbf{b}$ are orthogonal and $|\mathbf{t}| = |\mathbf{b}| = a_{11} = a_{22} = 1$, we conclude from Eq. 50 that the Weingarten matrix is symmetric, i.e. $b_1^2 = b_2^1$. In consequence, we may approximate the coefficients of the second fundamental form with respect to the vectors $(\mathbf{e}_i \cdot \mathbf{t})\mathbf{t}$ and $(\mathbf{e}_i \cdot \mathbf{b})\mathbf{b}$ as

$$\begin{aligned}
d_{(\mathbf{e}_i\cdot\mathbf{t})\mathbf{t}}\mathbf{n} = t_i d_\mathbf{t}\mathbf{n} &\approx ((\mathbf{n}_i - \mathbf{n})\cdot\mathbf{t})t_i = \Delta\mathbf{n}_{i,t}t_i \approx -b_1^1 t_i^2 - b_1^2 t_i b_i = b_{11}t_i^2 + b_{12}t_i b_i \\
d_{(\mathbf{e}_i\cdot\mathbf{b})\mathbf{b}}\mathbf{n} = b_i d_\mathbf{b}\mathbf{n} &\approx ((\mathbf{n}_i - \mathbf{n})\cdot\mathbf{b})b_i = \Delta\mathbf{n}_{i,b}b_i \approx -b_2^1 b_i t_i - b_2^2 b_i^2 = b_{12}b_i t_i + b_{22}b_i^2 \\
d_{(\mathbf{e}_i\cdot\mathbf{t})\mathbf{t}}\mathbf{n} = t_i d_\mathbf{t}\mathbf{n} &\approx ((\mathbf{n}_i - \mathbf{n})\cdot\mathbf{t})b_i = \Delta\mathbf{n}_{i,t}b_i \approx -b_1^1 t_i b_i - b_1^2 b_i^2 = b_{11}t_i b_i + b_{12}b_i^2 \\
d_{(\mathbf{e}_i\cdot\mathbf{b})\mathbf{b}}\mathbf{n} = b_i d_\mathbf{b}\mathbf{n} &\approx ((\mathbf{n}_i - \mathbf{n})\cdot\mathbf{b})t_i = \Delta\mathbf{n}_{i,b}t_i \approx -b_2^1 t_i^2 - b_2^2 t_i b_i = b_{12}t_i^2 + b_{22}t_i b_i
\end{aligned} \tag{91}$$

To make the computation robust under different distributions of triangles around a vertex, the same average approach for normal vectors (Section 2.5.1) is applied. We considere that each directional variation of the mesh normal contributes equally to the curvature tensor of v and sum the contributions of all n edges $\mathbf{e}_i$ incident to v

$$\begin{aligned}
\sum_{i=0}^{n-1}\Delta\mathbf{n}_{i,t}t_i &= b_{11}\sum_{i=0}^{n-1}t_i^2 + b_{12}\sum_{i=0}^{n-1}t_i b_i \\
\sum_{i=0}^{n-1}\Delta\mathbf{n}_{i,b}b_i &= b_{12}\sum_{i=0}^{n-1}b_i t_i + b_{22}\sum_{i=0}^{n-1}b_i^2. \\
\sum_{i=0}^{n-1}\Delta\mathbf{n}_{i,t}b_i + \sum_{i=0}^{n-1}\Delta\mathbf{n}_{i,b}t_i &= b_{11}\sum_{i=0}^{n-1}t_i b_i + b_{12}\sum_{i=0}^{n-1}(b_i^2 + t_i^2) + b_{22}\sum_{i=0}^{n-1}t_i b_i.
\end{aligned} \tag{92}$$

We apply the LU decomposition technique to solve the linear set of equations [36]. And, according to Eq. 45 the solutions b_1^1, $b_1^2 = b_2^1$, and b_2^2 build a matrix of dimension 2. Its eigenvectors and eigenvalues are, respectively, the estimated principal directions ($\mathbf{pd}_{(1)} = (pd^1_{(1)}, pd^2_{(1)})$ and $\mathbf{pd}_{(2)} = (pd^1_{(2)}, pd^2_{(2)})$) and the estimated principal curvatures (κ_1 and κ_2) of the mesh at v. With the principal directions we find its ambient coordinates

$$\begin{aligned}
\mathbf{d}_{(1)} &= pd^1_{(1)}\mathbf{t} + pd^2_{(1)}\mathbf{b} \\
\mathbf{d}_{(2)} &= pd^1_{(2)}\mathbf{t} + pd^2_{(2)}\mathbf{b}
\end{aligned} \tag{93}$$

and if we pass the basis vectors $\mathbf{t}$ and $\mathbf{b}$ to $\mathbf{d}_{(1)}$ and $\mathbf{d}_{(2)}$ we have in new coordinates (Eq. 52)

$$\begin{aligned}
\tilde{b}_{12} &= \tilde{a}_{12} = 0 \qquad (94) \\
\kappa_1 &= \frac{\tilde{b}_{11}}{\tilde{a}_{11}} \Longrightarrow \tilde{b}_{11} = \kappa_1\tilde{a}^{11} \\
\kappa_2 &= \frac{\tilde{b}_{22}}{\tilde{a}_{22}} \Longrightarrow \tilde{b}_{22} = \kappa_2\tilde{a}^{22}.
\end{aligned}$$

Once $\tilde{a}^{11} = \tilde{a}^{11} = 1$ (the principal directions are unitary), $\tilde{b}_{11} = \kappa_1$ and $\tilde{b}_{22} = \kappa_2$.

As our matrices have dimensions 2×2, we apply the Jacobi method to diagonalize them. Essentially the Jacobi method consists in zeroing the off-diagonal matrix elements by a series of plane rotations [36].

Nevertheless, it is worth remarking that the estimation accurary is highly dependent on the sampling resolution. When the sampling resolution is not sufficient for correctly capturing the variations of normal vectors as depicted in Figure 10, we may get completely wrong values. Once the way that a mesh is deformed cannot lead to more than one inflection point in our model, we propose to work around this problem in this work by detecting non-coplanarity in the vicinity of each vertex. Given a sample vertex $\mathbf{P}$ with the normal vector $\mathbf{n}_p$. We consider that an adjacent vertex $\mathbf{P}_i$ with the normal vector $\mathbf{n}_i$ is non-coplanar with respect to $\mathbf{P}$, if $\mathbf{n}_p \cdot \mathbf{n}_i \approx 1$ and $fabs(\mathbf{n}_p \cdot (\mathbf{P} - \mathbf{P}_i)) > \epsilon$. In this case, we further propose to use the average normal vector $\mathbf{N}$ of the normal vectors of the faces that are adjacent to the edge connecting the sampled adjacent vertices.

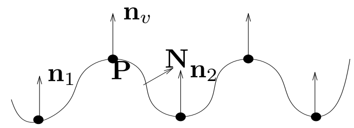

Figure 10: Low sampling resolution.

Another critical issue of our one-pass algorithm is the estimation of the components of the curvature tensor for the vertices on the boundary. This is because that the information on the symmetric half-side may be missed for inferring the bending behavior of a mesh in the vicinity of these vertices. We propose to round off this lack of information with the two-pass procedure proposed by Rusinkiewicz, slightly modified for accounting low sampling rate. Essentially, we apply Eq. 92 to compute the curvature tensors of the sub-divided neighboring faces F_j at the midpoint p_i of the adjacent edges of a boundary vertex, as shown in Figure 11, and transform the quantities to the same reference system, as explained in Section 2.5.4, before summing them up with the "Voronoi area" weights. Note that in this way, instead of normal vectors at adjacent vertices $\mathbf{n}_i$, we use the average vectors $\mathbf{aux}_i$ in the estimation of curvature tensors.

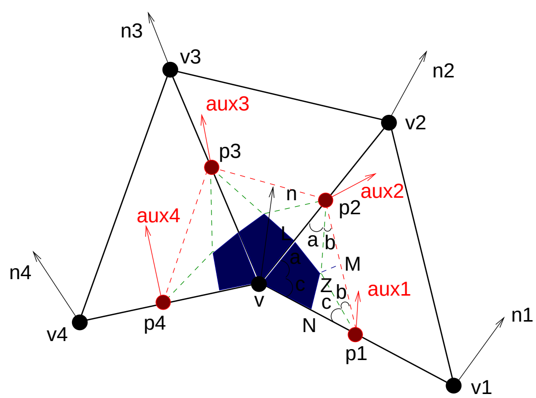

Figure 11: Curvature tensor at a boundary vertex as the "Voronoi area" weighted sum of the curvature tensors of its adjacent faces.

Any triangle can be circumscribed by a circle whose center Z divides a triangle in three sub-triangles as show the green dotted lines in Figure 11. The signed area of each sub-triangle can be obtained by multiplying the length of the corresponding edge with the height of the sub-triangle. This height can be computed from the circumradius ρ and the angle of each vertex. Without loss of generality, let's consider the case of the triangle vp_1p_2 in Figure 11, in which we have the edge lengths $e_2 = |p_2v|$, $e_1 = |p_1p_2|$ and $e_0 = |vp_1|$, the angles $\widehat{v} = a + c$, $\widehat{p_1} = b + c$, $\widehat{p_2} = a + b$ and $a + b + c = \frac{\pi}{2}$. With use of trigonometric identities

$$
\begin{aligned}
\sin(a) &= \sin(\frac{\pi}{2} - \widehat{p_1}) = \cos(\widehat{p_1}) \\
\sin(b) &= \sin(\frac{\pi}{2} - \widehat{v}) = \cos(\widehat{v}) \\
\sin(c) &= \sin(\frac{\pi}{2} - \widehat{p_2}) = \cos(\widehat{p_2}) \\
\cos(\widehat{v}) &= \frac{e_2^2 + e_0^2 - e_1^2}{2e_2e_0} \\
\cos(\widehat{p_1}) &= \frac{e_0^2 + e_1^2 - e_2^2}{2e_0e_1} \\
\cos(\widehat{p_2}) &= \frac{e_1^2 + e_2^2 - e_0^2}{2e_1e_2}
\end{aligned}
$$

we can get the signed height of each sub-triangle

$$|LZ| = \rho\sin(a) = \rho\cos(\widehat{p_1})$$

$$
\begin{aligned}
|MZ| &= \rho \sin(c) = \rho \cos(\widehat{p_2}) \\
|NZ| &= \rho \sin(b) = \rho \cos(\widehat{v}),
\end{aligned}
$$

and compute the signed areas of sub-triangles

$$
\begin{aligned}
A_{\triangle p_2 v Z} &= \frac{1}{2} e_2(\rho cos(\widehat{p_1})) = (\frac{1}{2}\rho)(e_2 cos(\widehat{p_1})) = (\frac{1}{2}\rho)\frac{e_2^2(e_0^2+e_1^2-e_2^2)}{2e_0e_1e_2} \\
A_{\triangle p_1 p_2 Z} &= \frac{1}{2} e_1(\rho cos(\widehat{v})) = (\frac{1}{2}\rho)(e_1 cos(\widehat{v})) = (\frac{1}{2}\rho)\frac{e_1^2(e_2^2+e_0^2-e_1^2)}{2e_2e_0e_1} \\
A_{\triangle v p_1 Z} &= \frac{1}{2} e_0(\rho cos(\widehat{p_2})) = (\frac{1}{2}\rho)(e_0 cos(\widehat{p_2})) = (\frac{1}{2}\rho)\frac{e_0^2(e_1^2+e_2^2-e_0^2)}{2e_1e_2e_0}. \qquad (95)
\end{aligned}
$$

Then, the total area of the triangle is

$$
\begin{aligned}
A_{\triangle} &= A_{\triangle v p_1 Z} + A_{\triangle p_1 p_2 Z} + A_{\triangle p_2 v Z} \\
&= \frac{\rho}{2}\frac{e_2^2(e_0^2+e_1^2-e_2^2)+e_1^2(e_2^2+e_0^2-e_1^2)+e_0^2(e_1^2+e_2^2-e_0^2)}{2e_0e_1e_2} \qquad (96)
\end{aligned}
$$

and the area ratios, or the barycentric coordinates (α, β, γ) of the circumcenter Z, can be given in terms of the triangle edge lengths

$$
\begin{aligned}
\alpha = \frac{A_{\triangle p_2 v Z}}{A_{\triangle}} &= \frac{e_2^2(e_0^2+e_1^2-e_2^2)}{e_2^2(e_0^2+e_1^2-e_2^2)+e_1^2(e_2^2+e_0^2-e_1^2)+e_0^2(e_1^2+e_2^2-e_0^2)}, \\
\beta = \frac{A_{\triangle p_1 p_2 Z}}{A_{\triangle}} &= \frac{e_1^2(e_2^2+e_0^2-e_1^2)}{e_2^2(e_0^2+e_1^2-e_2^2)+e_1^2(e_2^2+e_0^2-e_1^2)+e_0^2(e_1^2+e_2^2-e_0^2)}, \\
\gamma = \frac{A_{\triangle v p_1 Z}}{A_{\triangle}} &= \frac{e_0^2(e_1^2+e_2^2-e_0^2)}{e_2^2(e_0^2+e_1^2-e_2^2)+e_1^2(e_2^2+e_0^2-e_1^2)+e_0^2(e_1^2+e_2^2-e_0^2)}.
\end{aligned}
$$

The sum of the half of the areas of the sub-triangles that are adjacent to each vertex delivers its Voronoi area with respect to the triangle. For example, for the face $\triangle v p_1 p_2$ the Voronoi area of v is $\frac{1}{2}A_{\triangle p_2 v Z} + \frac{1}{2}A_{\triangle v p_1 Z}$.

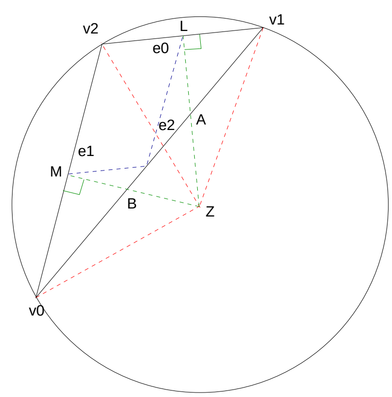

Figure 12: Obtuse tringle.

When the triangle is obtuse, the signed areas may be negative. Without loss of generality, let's use Figure 12 to illustrate two different proposals for the Voronoi areas. Rusinkiewicz considers in his implementation [38] that the areas of $\triangle v_1 L A$ and $\triangle v_0 B M$ are, respectively, the (negative) Voronoi areas relative to the acute angles at vertices v_1 and v_0

$$
\begin{aligned}
A_{\triangle v_1 L A} &= -\frac{1}{2}\frac{e_0}{2}(\frac{\frac{e_0}{2}}{\cos(\widehat{v_1})}\sin(\widehat{v_1})) \\
A_{\triangle v_0 B M} &= -\frac{1}{2}\frac{e_1}{2}(\frac{\frac{e_1}{2}}{\cos(\widehat{v_0})}\sin(\widehat{v_0})). \qquad (97)
\end{aligned}
$$

As the area of $\triangle v_0 v_1 v_2$ is

$$
A_{\triangle v_0 v_1 v_2} = \frac{1}{2} e_0 e_2 \sin(\widehat{v_1}) = \frac{1}{2} e_1 e_2 \sin(\widehat{v_0}),
$$

we can write

$$\sin(\widehat{v_1}) = \frac{2A_{\triangle v_0 v_1 v_2}}{e_0 e_2}$$
$$\sin(\widehat{v_0}) = \frac{2A_{\triangle v_0 v_1 v_2}}{e_1 e_2}.$$

Replacing them in Eq. 97, we have

$$A_{\triangle v_1 LA} = -\frac{1}{2}\frac{e_0^2}{2}(\frac{A_{\triangle v_0 v_1 v_2}}{e_0 e_2 \cos(\widehat{v_1})}) = -\frac{1}{4}e_0^2(\frac{A_{\triangle v_0 v_1 v_2}}{e_0 \cdot e_2})$$
$$A_{\triangle v_0 BM} = -\frac{1}{2}\frac{e_1^2}{2}(\frac{A_{\triangle v_0 v_1 v_2}}{e_1 e_2 \cos(\widehat{v_0})}) = -\frac{1}{4}e_1^2(\frac{A_{\triangle v_0 v_1 v_2}}{e_1 \cdot e_2}).$$

The Voronoi area of the obtuse angle is such one that together with the areas of other sub-triangles make up $A_{\triangle v_0 v_1 v_2}$

$$A_{\triangle v_0 v_1 v_2} = A_{\triangle v_1 LA} + A_{\triangle v_0 BM} + A_{BALM} \Longrightarrow A_{BALM} = A_{\triangle v_0 v_1 v_2} - A_{\triangle v_1 LA} - A_{\triangle v_0 BM}$$

For avoiding negative weighting, Meyer et al. proposes to simply sub-divide the triangle by connecting the midpoints of the edges, as the blue dotted lines in Figure 12 [32]. The Voronoi areas are $\frac{A_{\triangle v_0 v_1 v_2}}{4}$ and $\frac{A_{\triangle v_0 v_1 v_2}}{2}$ for the acute and the obtuse angles, respectively.

2.5.3 Coefficients of a Metric Tensor

Knowing that the components of the metric tensor in any basis of vectors, or frame, are given by the dot products of these vectors 10, we may choose any pair of non-normalized, non-collinear estimated tangent vectors and use them to compute the components of a metric tensor.

We propose to estimate a pair of tangent vectors for each vertex v employing the estimated principal directions $\mathbf{d}_{(1)}$ and $\mathbf{d}_{(2)}$ (Eq. 93). Observe that their cross product gives us the normal **new_n** of the estimated tangent plane at v. We decompose all the edges $\mathbf{e}_i = v_i - v$ adjacent to v (Figure 9) in two components: one parallel to the normal **new_n** and one orthogonal to $\mathbf{n}_v$. This is obtained by subtracting the projection of $\mathbf{e}_i$ on **new_n** from itself

$$\mathbf{te}_i = \mathbf{e}_i - (\mathbf{e}_i \cdot \mathbf{new\_n})\mathbf{new\_n}. \tag{98}$$

Among $\mathbf{te}_i$ we select the vector with maximum length and use it to define

$$\mathbf{new\_t} = |\mathbf{e}_i|\frac{\mathbf{te}_i}{|\mathbf{te}_i|}.$$

Next, we determine among $\mathbf{te}_j, j \neq i$, the one that has the smallest non-null dot product with **new_t**. We use the notation **new_b** to represent this vector, as shown in Figure 13. Observe that the vectors (**new_t**, **new_b**, **new_n**) build a reference system and this reference system has the advantage of approximately embedding the distances between adjacent surface samples in their basis vectors because the components of the metric tensors are

$$a_{11} = \mathbf{new\_t} \cdot \mathbf{new\_t} \qquad a_{22} = \mathbf{new\_b} \cdot \mathbf{new\_b} \qquad a_{12} = \mathbf{new\_t} \cdot \mathbf{new\_b}.$$

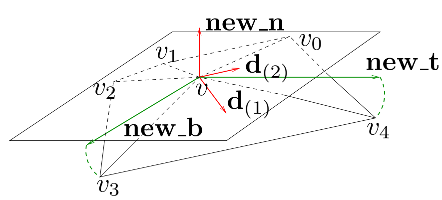

Figure 13: Principal directions (in red) and estimated basis vectors with respect to the adjacent edges (in green).

2.5.4 Coordinate Transformations

In our application it is desirable to express the curvature tensor in the same reference system as the metric tensor, namely the one presented in Section 2.5.3. Since curvature tensors are transformed under covariant transformation, we may apply Eq. 16 to transform curvature tensors from the basis formed by the principal directions in ambient coordinates $(\mathbf{d}_{(1)}\mathbf{d}_{(2)}, \mathbf{n}_{dp})$, where $\mathbf{n}_{dp} = \frac{\mathbf{d}_{(1)} \times \mathbf{d}_{(2)}}{|\mathbf{d}_{(1)} \times \mathbf{d}_{(2)}|}$, into the frame $(\mathbf{new\_t}, \mathbf{new\_b}, \mathbf{new\_n})$

$$\begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \rightarrow \begin{bmatrix} -b_1^1 & -b_1^2 \\ -b_2^1 & -b_2^2 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{b_{12}a_{12} - b_{11}a_{22}}{a_{11}a_{22} - a_{12}^2} & \frac{b_{11}a_{12} - b_{12}a_{11}}{a_{11}a_{22} - a_{12}^2} \\ \frac{b_{22}a_{12} - b_{12}a_{22}}{a_{11}a_{22} - a_{12}^2} & \frac{b_{12}a_{12} - b_{22}a_{11}}{a_{11}a_{22} - a_{12}^2} \end{bmatrix} \rightarrow \begin{bmatrix} -b_{11} & -b_{12} \\ -b_{12} & -b_{22} \end{bmatrix}$$

If $\mathbf{n}_{dp}$ and $\mathbf{n}$ are equal, $(\mathbf{d}_{(1)}, \mathbf{d}_{(2)})$ and $(\mathbf{new\_t}, \mathbf{new\_b})$ are coplanar. We may adopt the following discrete version of the variation rates of the coordinates dp^1 and dp^2 with regard to the coordinates t and b to find the curvature tensors in terms of $(\mathbf{new\_t}, \mathbf{new\_b}, \mathbf{n})$

$$\begin{aligned} \frac{\Delta \mathbf{d}_{(1)}}{\Delta \mathbf{new\_t}} &= \mathbf{d}_{(1)} \cdot \mathbf{new\_t} \\ \frac{\Delta \mathbf{d}_{(1)}}{\Delta \mathbf{new\_b}} &= \mathbf{d}_{(1)} \cdot \mathbf{new\_b} \\ \frac{\Delta \mathbf{d}_{(2)}}{\Delta \mathbf{new\_t}} &= \mathbf{d}_{(2)} \cdot \mathbf{new\_t} \\ \frac{\Delta \mathbf{d}_{(2)}}{\Delta \mathbf{new\_b}} &= \mathbf{d}_{(2)} \cdot \mathbf{new\_b}. \end{aligned} \tag{99}$$

When $(\mathbf{d}_{(1)}, \mathbf{d}_{(2)})$ and $(\mathbf{t}, \mathbf{b})$ are not coplanar, Rusinkiewicz observed in [39] that one cannot simply project the axes. He proposed to rotate the normal vector $\mathbf{n}_{dp}$ such that it aligns with $\mathbf{n}$ before applying Eq. 99. Let's take a look at his proposal.

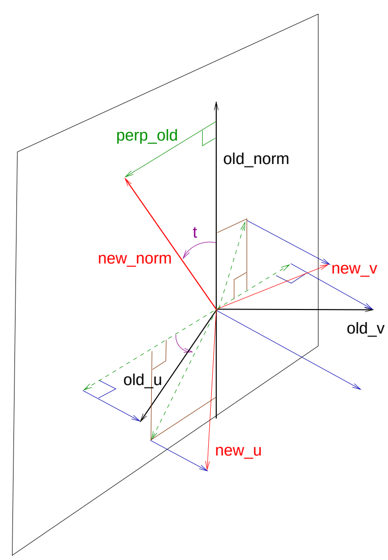

Figure 14: Rotation of an old reference system (in black) towards a new reference (in red), such that the normal vectors become collinear.

Given an old reference system $(\mathbf{old}_u, \mathbf{old}_v, \mathbf{old}_{norm})$ and a new reference system $(\mathbf{new}_u, \mathbf{new}_v, \mathbf{new}_{norm})$. When $\mathbf{old}_{norm}$ is rotated towards to $\mathbf{new}_{norm}$ such that they become collinear as illustrates Figure 14, $\mathbf{new}_{norm}$ can be described as the combination of two orthogonal vectors

$$\begin{aligned} \mathbf{new}_{norm} &= \mathbf{perp}_{old} + \cos(\theta)\mathbf{new}_{norm} \\ &= \mathbf{perp}_{old} + (\mathbf{old}_{norm} \cdot \mathbf{new}_{norm})\mathbf{old}_{norm}. \end{aligned}$$

Since $|\mathbf{perp}_{old}| = \sin(\theta) = \sqrt{1 - ndot^2}$, where $ndot = \cos(\theta) = \mathbf{old}_{norm} \cdot \mathbf{new}_{norm}$, $\mathbf{perp}_{old}$ can be normalized by simply dividing it with $\sqrt{1 - ndot^2}$. Rusinkiewicz observed that if he decomposed

$\mathbf{old}_u/\mathbf{old}_u$ into two orthogonal vectors, one in the direction of the normal vector of the plane $\mathcal{P}$ containing $\mathbf{old}_{norm}$ and $\mathbf{new}_{norm}$, drawn in blue in Figure 14, and another in the direction of $\frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}}$, drawn in green in Figure 14, only the components on $\mathcal{P}$ are rotated by the same angle θ that $\mathbf{old}_{norm}$ was rotated. To get these components, he projected $\mathbf{old}_u$ on $\frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}}$ and got the length of each vector-component

$$|Proj_{\mathcal{P}}(\mathbf{old}_u)| = \mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}}$$

After rotated by the angle θ, this vector on $\mathcal{P}$ became

$$(\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})\cos(\theta)\frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}} - (\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})\sin(\theta)\mathbf{old}_{norm}$$

which is equivalent to

$$(\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})ndot\frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}} - (\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})\sqrt{1-ndot^2}\mathbf{old}_{norm} \tag{100}$$

The new vector $\mathbf{new}_u$ can then be obtained by subtracting the vector $(\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})\frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}}$ and adding Eq. 100 as follows

$$\begin{aligned}
\mathbf{new}_u &= \mathbf{old}_u - (\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})\frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}} + \\
&\quad +((\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})ndot\frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}} - (\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})\sqrt{1-ndot^2}\mathbf{old}_{norm}) \\
&= \mathbf{old}_u - ((\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})\frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}} - \\
&\quad -(\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})ndot\frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}} + (\mathbf{old}_u \cdot \frac{\mathbf{perp}_{old}}{\sqrt{1-ndot^2}})\sqrt{1-ndot^2}\mathbf{old}_{norm}) \\
&= \mathbf{old}_u - (\mathbf{old}_u \cdot \mathbf{perp}_{old})(\frac{1-ndot}{1-ndot^2}\mathbf{perp}_{old} + \mathbf{old}_{norm}) \\
&= \mathbf{old}_u - (\mathbf{old}_u \cdot \mathbf{perp}_{old})\frac{1}{1+ndot}(\mathbf{new}_{norm} - ndot\ \mathbf{old}_{norm} + (1+ndot)\ \mathbf{old}_{norm}) \\
&= \mathbf{old}_u - (\mathbf{old}_u \cdot \mathbf{perp}_{old})\frac{1}{1+ndot}(\mathbf{new}_{norm} + \mathbf{old}_{norm}).
\end{aligned} \tag{101}$$

The new vector $\mathbf{new}_u$ can be analogously derived

$$\mathbf{new}_v = \mathbf{old}_v - (\mathbf{old}_v \cdot \mathbf{perp}_{old})\frac{1}{1+ndot}(\mathbf{new}_{norm} + \mathbf{old}_{norm}). \tag{102}$$

We devised an alternative procedure that extracts the derivatives of the coordinates dp^1 and dp^2 with respect to the coordinates t and b from the Jacobian matrix $J(new\_t, new\_b, new\_n)$

$$J(new\_t, new\_b, new\_n) = \begin{bmatrix} \frac{\partial dp^1}{\partial t} & \frac{\partial dp^1}{\partial b} & \frac{\partial dp^1}{\partial n} \\ \frac{\partial dp^2}{\partial t} & \frac{\partial dp^2}{\partial b} & \frac{\partial dp^2}{\partial n} \\ \frac{\partial dp^3}{\partial t} & \frac{\partial dp^3}{\partial b} & \frac{\partial dp^3}{\partial n} \end{bmatrix}$$

This matrix transforms the basis vector $\tilde{B} = (\mathbf{d}_{(1)}, \mathbf{d}_{(2)}, \mathbf{n}_{dp})$ to $B = (\mathbf{new\_t}, \mathbf{new\_b}, \mathbf{new\_n})$

$$\begin{aligned}
B &= \tilde{B}J(new_t, new_b, new_n) \Longrightarrow J(new_t, new_b, new_n) = \\
&\quad \tilde{B}^{-1}B = \begin{bmatrix} \mathbf{d}_{(1)} & \mathbf{d}_{(2)} & \mathbf{n}_{dp} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{new_t} & \mathbf{new_b} & \mathbf{new_n} \end{bmatrix}.
\end{aligned} \tag{103}$$

For quantities that transform contravariantly, as in Eq. 26, we should take the Jacobian matrix of the inverse function from $B = (\mathbf{new\_t}, \mathbf{new\_b}, \mathbf{new\_n})$ to $\tilde{B} = (\mathbf{d}_{(1)}, \mathbf{d}_{(2)}, \mathbf{n}_{dp})$, namely

$$\begin{aligned}
\tilde{B} &= BJ(d_1, d_2, n_{dp}) \Longrightarrow J(d_1, d_2, n_{dp}) = \\
&\quad B^{-1}\tilde{B} = [B^{-1})^{-1}(\tilde{B}]^{-1} = [\tilde{B}^{-1}B]^{-1} = [J(new_t, new_b, new_n)]^{-1},
\end{aligned} \tag{104}$$

which is expected from the inverse function theorem.

2.5.5 Christoffel Symbols

In this work we need Christoffel symbols for compensating the basis changes when we evaluate the difference between two quantities given in their own local reference system, as shown in Section 2.4.4. In Section 2.4.2 we present several equivalent formulations to express them. Because of the number of required terms, we propose in this work to take the discrete form of Eq. 77 for estimating these geometric quantities at each sample of a surface mesh of arbitrary topology.

Our starting point are the local reference systems estimated with use of the procedure described in Section 2.5.3. Figure 15 illustrates the local reference systems in the adjacency of a surface sample $\mathcal{P}(t,b)$. They are $\mathbf{r}_t(t+\Delta t,b)$ and $\mathbf{r}_b(t+\Delta t,b)$ at the sample $\mathcal{P}(t+\Delta t,b)$, and $\mathbf{r}_t(t,b+\Delta b)$ and $\mathbf{r}_b(t,b+\Delta b)$ at the sample $\mathcal{P}(t,b+\Delta b)$. Observe that these first derivatives are given with respect to the ambient space and we may apply a classical finite differencing scheme for estimating partial second derivatives on discrete data: the forward, the backward or the central difference scheme.

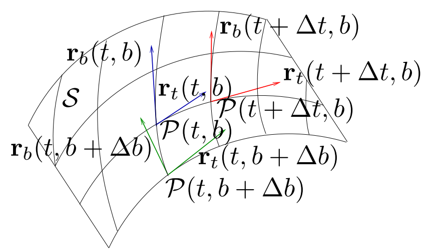

Figure 15: Adjacent local reference systems.

For example, if we consider that our parametrization is by arc length, the second derivatives of $\mathbf{r}_t$ with respect to t in the forward difference scheme is

$$\mathbf{r}_{tt} \approx \frac{\mathbf{r}_t(t+\Delta t,b)-\mathbf{r}_t(t,b)}{\Delta t} \approx \frac{\mathbf{r}_t(t+\Delta t,b)-\mathbf{r}_t(t,b)}{\mathcal{P}(t+\Delta t,b)-\mathcal{P}(t,b)}, \tag{105}$$

the backward difference scheme has the form

$$\mathbf{r}_{tt} \approx \frac{\mathbf{r}_t(t,b)-\mathbf{r}_t(t-\Delta t,b)}{\Delta t} \approx \frac{\mathbf{r}_t(t,b)-\mathbf{r}_t(t-\Delta t,b)}{\mathcal{P}(t,b)-\mathcal{P}(t-\Delta t,b)}, \tag{106}$$

and the central difference scheme considers both adjacent samples

$$\mathbf{r}_{tt} \approx \frac{\mathbf{r}_t(t+\Delta t,b)-\mathbf{r}_t(t-\Delta t,b)}{\Delta t} \approx \frac{\mathbf{r}_t(t+\Delta t,b)-\mathbf{r}_t(t-\Delta t,b)}{\mathcal{P}(t+\Delta t,b)-\mathcal{P}(t-\Delta t,b)}. \tag{107}$$

Since all finite differencing schemes are based on a Taylor expansion of the function to be differentiated and they only vary on the terms that are truncated, our choice is based on the approximation error. It is known that the approximation error for central differences is of higher order compared with forward or backward differences [36] and that the central scheme is the most common approach for derivative estimation in Computer Graphics. It is thus better to apply the central difference scheme whenever it is possible and to compute the Christoffel symbols of the second kind as the scalar products of the first derivatives of the basis vectors and the contravariant basis vectors.

$$\begin{array}{lll} \Gamma^1_{11} = \mathbf{r}_{tt}\cdot\mathbf{r}^t & \Gamma^1_{12} = \Gamma^1_{21} = \mathbf{r}_{tb}\cdot\mathbf{r}^t & \Gamma^1_{22} = \mathbf{r}_{bb}\cdot\mathbf{r}^t \\ \Gamma^2_{11} = \mathbf{r}_{tt}\cdot\mathbf{r}^b & \Gamma^2_{12} = \Gamma^2_{21} = \mathbf{r}_{tb}\cdot\mathbf{r}^b & \Gamma^2_{22} = \mathbf{r}_{bb}\cdot\mathbf{r}^b \end{array} \tag{108}$$

where the contravariant basis vectors can be determined with use of Eq. 21 and the necessary components of the contravariant metric tensor can be get from inverting the covariant metric tensor

$$\begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1}$$

Alternatively, we use a differencing scheme of Eq. 33 and Eq. 36 to estimate the Christoffel symbols

$$\Gamma^\gamma_{\alpha\beta} \quad = \quad a^{1\gamma}\Gamma_{\alpha\beta 1} + a^{2\gamma}\Gamma_{\alpha\beta 2}$$

$$
\begin{aligned}
&= (a^{1\gamma}(\frac{\partial a_{\beta 1}}{\partial coord(\alpha)} + \frac{\partial a_{1\alpha}}{\partial coord(\beta)} - \frac{\partial a_{\alpha\beta}}{\partial u}) + a^{2\gamma}(\frac{\partial a_{\beta 2}}{\partial coord(\alpha)} + \frac{\partial a_{2\alpha}}{\partial coord(\beta)} - \frac{\partial a_{\alpha\beta}}{\partial v})) \\
&= (a^{1\gamma}(\frac{\Delta a_{\beta 1}}{\Delta coord(\alpha)} + \frac{\Delta a_{1\alpha}}{\Delta coord(\beta)} - \frac{\Delta a_{\alpha\beta}}{\Delta u}) + a^{2\gamma}(\frac{\Delta a_{\beta 2}}{\Delta coord(\alpha)} + \frac{\Delta a_{2\alpha}}{\Delta coord(\beta)} - \frac{\Delta a_{\alpha\beta}}{\Delta v}))
\end{aligned}
\tag{109}
$$

where $coord(1) = u$ and $coord(2) = v$

2.5.6 Derivatives of Curvature Tensor

Particularly for the derivatives of curvature tensors, we have the expressions given in Eq. 68. Gravesen and Ungstrup showed us that only four among eight coefficients are distinct [22]. Rusinkiewicz further observed that a simple extension to the curvature-estimation algorithm can be used to estimate "derivatives of curvature": just as curvatures are estimated per-vertex by considering the differences in normals along the edges, we may use the differences in the normal curvature along the edges $\mathbf{e}_i$ adjacent to each vertex v [39]. His procedure has the advantage that considers all known variations around v.

Defining an orthonormal reference system $(\mathbf{t}, \mathbf{b}, \mathbf{n})$ as in Section 2.5.2 at each vertex, the directional derivatives of the coefficients of the curvature tensor along the projections of an edge $\mathbf{e}_i$ on the tangential plane can be approximated as follows

$$
\begin{aligned}
d_{(\mathbf{e}_i\cdot\mathbf{t})\mathbf{t}} b_{11} &= t_i db_{11} \approx t_i \Delta b_{11} \approx t_i(b_{11|t} t_i + b_{11|b} b_i) \\
d_{(\mathbf{e}_i\cdot\mathbf{b})\mathbf{b}} b_{11} &= b_i db_{11} \approx b_i \Delta b_{11} \approx b_i(b_{11|t} t_i + b_{11|b} b_i) \\
d_{(\mathbf{e}_i\cdot\mathbf{t})\mathbf{t}} b_{12} &= t_i db_{12} \approx t_i \Delta b_{12} \approx t_i(b_{12|t} t_i + b_{12|b} b_i) \\
d_{(\mathbf{e}_i\cdot\mathbf{b})\mathbf{b}} b_{12} &= b_i db_{12} \approx b_i \Delta b_{12} \approx b_i(b_{12|t} t_i + b_{12|b} b_i) \\
d_{(\mathbf{e}_i\cdot\mathbf{t})\mathbf{t}} b_{21} &= t_i db_{21} \approx t_i \Delta b_{21} \approx t_i(b_{21|t} t_i + b_{21|b} b_i) \\
d_{(\mathbf{e}_i\cdot\mathbf{b})\mathbf{b}} b_{21} &= b_i db_{21} \approx b_i \Delta b_{21} \approx b_i(b_{21|t} t_i + b_{21|b} b_i) \\
d_{(\mathbf{e}_i\cdot\mathbf{t})\mathbf{t}} b_{22} &= t_i db_{22} \approx t_i \Delta b_{22} \approx t_i(b_{22|t} t_i + b_{22|b} b_i) \\
d_{(\mathbf{e}_i\cdot\mathbf{b})\mathbf{b}} b_{22} &= b_i db_{22} \approx b_i \Delta b_{22} \approx b_i(b_{22|t} t_i + b_{22|b} b_i).
\end{aligned}
\tag{110}
$$

As we have four unknowns, we compact these expressions into a system of 4 linear equations

$$
\begin{aligned}
t_i \Delta b_{11} &\approx t_i^2 b_{11|t} + t_i b_i b_{11|b} = t_i^2 b_{11|t} + t_i b_i b_{12|t} \\
b_i \Delta b_{11} + t_i \Delta b_{12} + t_i \Delta b_{21} = b_i \Delta b_{11} + 2 t_i \Delta b_{12} &\approx b_i(b_{11|t} t_i + b_{11|b} b_i) + t_i(b_{21|t} t_i + b_{21|b} b_i) + t_i(b_{12|t} t_i + b_{12|b} b_i) \\
&= b_i t_i b_{11|t} + b_i^2 b_{12|t} + t_i^2 b_{12|t} + t_i b_i b_{12|b} + t_i^2 b_{12|t} + b_i t_i b_{12|b} \\
&= b_i t_i b_{11|t} + (2t_i^2 + b_i^2) b_{12|t} + 2 t_i b_i b_{12|b} \\
t_i \Delta b_{22} + b_i \Delta b_{12} + b_i \Delta b_{21} = t_i \Delta b_{22} + 2 b_i \Delta b_{12} &\approx b_i(b_{12|t} t_i + b_{12|b} b_i) + b_i(b_{21|t} t_i + b_{21|b} b_i) + t_i(b_{22|t} t_i + b_{22|b} b_i) \\
&= b_i t_i b_{12|t} + b_i^2 b_{12|b} + b_i t_i b_{12|t} + b_i^2 b_{12|b} + t_i^2 b_{12|b} + b_i t_i b_{22|b} \\
&= 2 b_i t_i b_{12|t} + (t_i^2 + 2b_i^2) b_{12|b} + b_i t_i b_{22|b} \\
b_i \Delta b_{22} &\approx b_i t_i b_{22|t} + b_i^2 b_{22|b}
\end{aligned}
$$

Similarly to the estimation of curvature tensors, Rusinkiewicz proposed the following summations for attenuating possible unwanted biases

$$
\begin{aligned}
\sum_{i=0}^{n-1} \Delta b_{11,i} t_i &\approx b_{11|t} \sum_{i=0}^{n-1} t_i^2 + b_{12|t} \sum_{i=0}^{n-1} t_i b_i \\
\sum_{i=0}^{n-1} \Delta b_{11,i} b_i + 2 \sum_{i=0}^{n-1} \Delta b_{12,i} t_i &\approx b_{11|t} \sum_{i=0}^{n-1} t_i b_i + b_{12|t} \sum_{i=0}^{n-1} (2t_i^2 + b_i^2) + b_{12|b} \sum_{i=0}^{n-1} (2 t_i b_i) \\
2 \sum_{i=0}^{n-1} \Delta b_{21,i} b_i + \sum_{i=0}^{n-1} \Delta b_{22,i} t_i &\approx b_{12|t} \sum_{i=0}^{n-1} (2 t_i b_i) + b_{12|b} \sum_{i=0}^{n-1} (t_i^2 + 2b_i^2) + b_{22|b} \sum_{i=0}^{n-1} t_i b_i \\
\sum_{i=0}^{n-1} \Delta b_{22,i} b_i &\approx b_{12|b} \sum_{i=0}^{n-1} t_i b_i + b_{22|b} \sum_{i=0}^{n-1} b_i^2.
\end{aligned}
$$

The LU decomposition technique is applied to solve this system of linear equations [36].

It remains to show how to compute $\Delta b_{11,i}$, $\Delta b_{12,i} = \Delta b_{21,i}$, and $\Delta b_{22,i}$. According to the proposal presented by Rusinkiewicz [39], we perform this task in two steps. At the first step, we estimate the curvature tensor at each vertex using the procedure presented in Section 2.5.2. Then, for each vertex v we compute the differences between the coefficients of its curvature tensor, b_{11}, b_{12} and b_{22}, and the ones of its adjacent vertex v_i, $b_{11,i}$, $b_{12,i}$ and $b_{22,i}$,

$$\begin{aligned}\Delta b_{11,i} &= b_{11,i} - b_{11} \\ \Delta b_{12,i} &= b_{12,i} - b_{12} \\ \Delta b_{22,i} &= b_{22,i} - b_{22}.\end{aligned}$$

Nevertheless, the procedure presented in Section 2.5.2 estimates the curvature tensor with regard to a reference $(\mathbf{t}, \mathbf{b}, \mathbf{n})$ chosen arbitrarily and independently at each vertex v_i. There is no reference connection between them. To make the subtraction physically valid, we should transform covariantly, with the use Eq. 15, the coefficients of the tensor curvature from the reference fixed at v_i, $(\mathbf{t}_i, \mathbf{b}_i, \mathbf{n}_i)$ to the reference at v, $(\mathbf{t}, \mathbf{b}, \mathbf{n})$. The derivative terms in Eq. 15 are computed by the algorithm explained in Section 2.5.4.

2.5.7 Gaussian and Mean Curvatures

The Gaussian curvature and the absolute value of the mean curvature are invariant under coordinate transformation. After computing the principal curvatures (Section 2.5.2), Eq. 70 and Eq. 71 may be used to determine the Gaussian and the mean curvatures, respectively.

Special attention must be paid to the sign of the mean curvature, which is dependent on the parametrization. In most applications, it is desired that the Jacobian of two coordinate neighborhoods is positive. For this reason, additional Jacobian test is performed to coherently adjust the sign of the mean curvatures.

3 Related Work

It is a fact that the bending measures are crucial for enhancing the realism in the appearance of simulated clothes. Recently, two problems have been investigated: which quantities can be used as bending measures and how to appropriately formulate them for numerical implementations. In order to not be extensive, in this section we will refer to some pioneering works that allow us to illustrate the efforts in the last decades.

Physically based models are acknowledged to be the most promising ones for producing natural appearance to the clothes in motion. They consider that the cloth dynamics are ruled by the partial differential equilibrium equation at each point $\mathbf{r}$ [15]

$$\begin{aligned}\mu\frac{\partial^2\mathbf{r}}{\partial t^2} + \varrho\frac{\partial\mathbf{r}}{\partial t} + \frac{\delta\mathcal{A}(\mathbf{r},t)}{\delta\mathbf{r}} &= \mu\frac{\partial^2\mathbf{r}}{\partial t^2} + \varrho\frac{\partial\mathbf{r}}{\partial t} + \mathbf{K}(\mathbf{r},t)\mathbf{r}(t) \\ &= \mu\mathbf{F}(\mathbf{r},t)\ , \end{aligned} \qquad (111)$$

where μ is the mass density (mass per unit area), ϱ is the coefficient of the damping forces that counteract the intrinsic textile frictions and the external fluid frictions, $\mathbf{F}$ denotes the total contribution of external forces per unit mass on $\mathbf{r}$ and the term $\frac{\delta\mathcal{A}(\mathbf{r},t)}{\delta\mathbf{r}}$ corresponds to the internal energy per unit area governing the cloth's flexible appearance. The parameter $\mathbf{K}(\mathbf{r},t)$ is usually called the cloth's stiffness. Eq. 111 may further be rewritten as the basic Newton's law if we consider the friction forces as a pure external or internal force [37]

$$\begin{aligned}\mu\frac{\partial^2\mathbf{r}}{\partial t^2} &= -\mathbf{K}(\mathbf{r},t)\mathbf{r}(t) + (\mu\mathbf{F}(\mathbf{r},t) - \varrho\frac{\partial\mathbf{r}}{\partial t}) \\ &= -(\mathbf{K}(\mathbf{r},t)\mathbf{r}(t) + \varrho\frac{\partial\mathbf{r}}{\partial t}) + \mu\mathbf{F}(\mathbf{r},t).\end{aligned} \qquad (112)$$

Conjecturing that the stability and efficiency of a cloth simulation system may rely on the numerical solution scheme for Eq. 111 or Eq. 112, a remarkable amount of effort has been spent on it. The explicit integration methods reigned in '90s [33, 37, 45, 16]. Only at the end of 90s', Baraff and Witkin demonstrate in [2] the superiority of implicit (backward) numerical integration scheme in comparison to the explicit (forward) ones. This is because in an implicit scheme the new velocities $\dot{\mathbf{r}}(t + \Delta t)$ are computed in terms of the force conditions at $t + \Delta t$ instead of t. For alleviating the time-consuming computation of a large non-symmetric sparse linear system, Baraff and Witkin also develop a modified conjugate gradient method. The results are so promising that, since then, a series of works has been devoted to improve the implicit framework [14, 28, 11, 8]. Nevertheless, simply employing implicit integration method cannot overcome all instability problems that hinder stable responses [11]. Current research stresses that, because of its very particular behavior (comparatively resistant to stretching and shearing and permissive to bending), there is still room for improving the cloth's model, specially its bending behavior [24, 44, 5].

There are essentially two approaches for modeling the microstructure of fabrics: the particle or mass-spring paradigm, in which a fabric is considered as a collection of material points held together by structural, shear and flexion springs for simulating its material mechanics properties [6, 7, 37, 2, 14, 11, 8, 35, 26]; and the continuum mechanics based technique, in which a fabric is regarded as a continuous media to which the nonlinear thin shell or thin plate theory is applied for analyzing its stretching, shearing, and bending/flexural behavior [19, 15, 10, 17, 46, 24, 25, 47, 44, 5]. It is worth remarking that, after spatial and time finite differentiations, the particle and the continuum approaches have similar ordinary differential formulations [14]. They differ essentially in the *constitutive equations* [2], which are responsible for the internal force $\mathbf{K}(\mathbf{r}, t)\mathbf{r}(t)$ due to the cloth deformation.

In the particle or mass-spring approach, the internal force is modeled as the resultant of the tensions of the springs linking a point $\mathcal{P}_i$ to all its neighboring points $\mathcal{P}_j$. Breen et al. propose an angular expression for the bending measures [6]. They observe that a single thread can bend "out-of-plane" around crossing threads and describe this phenomenon by the angle formed between each set of three adjacent crossing nodes (or particles) in a rectangular mesh. More accurate control in the bending shape is later proposed by Volino et al. [45]. They used the angle and the radius of curvature. To improve the realism, Provot introduces in [37] the flexion springs to implicitly control the angular variations and formulates the constitutive equations solely in terms of $\mathbf{r}$ and its derivatives. Unlike the differential geometry approach, it is not sure that the angles can actually reflect the shape of the surface in the vicinity of $\mathbf{r}$ and, therefore, the accuracy of the internal force response. To lessen unrealistic residues due to the deviated force directions, Provot also proposes to define damping forces for dissipating them. The results are so convincing that Baraff and Witkin reformulate the angular expression in terms of the dihedral angles [2]. Detailed derivation of their expression may be found in [8]. As the previous angular based algorithms, they still need the fictitious damping forces to attenuate the unrealistic residual forces. Simply neglecting the in-plane stretching forces or overlooking the interdependency of the angular and the linear variations is the main flaw of these works.

The problem concerning the residual forces has been carefully analyzed by Choi and Ko who conclude that it stems from the fact that the existing bending model cannot appropriately deal with the cloth's behavior under compression [11]. As a solution, they propose to separately treat the "distension" (type 1 interaction) and the "compression" (type 2 interaction) cases, and provide a way to predict the post-buckling (bending) responses in terms of the arc length, the distances between the particles, and the bending stiffness. In essence, they turn to the point that Chen and Govindaraj have already emphasized in [10]: the link between the stretching and bending measures are crucial in cloth's modeling. From the theory of a Cosserat surface, which is founded on continuum mechanics, this link may be represented by the product of the (current) stretching and (current) bending measures. For avoiding complex expression, the initial bending measures are instead used in [10]. In this work we apply the Weingarten compatibility condition to naturally rule the in-plane and out-of-plane deformations.

Departing completely from the mass-spring approach, the internal force in the continuum mechanics is expressed as a function of the variation of $\mathcal{A}(\mathbf{r}, t)$ to the stretching measures ε and to the bending measures κ:

$$K(\mathbf{r}, t)\mathbf{r}(t) = \mu \frac{\delta \mathcal{A}(\mathbf{r}, t)}{\delta \mathbf{r}(t)} = \frac{\partial \mathcal{A}(\mathbf{r}, t)}{\partial \varepsilon(t)} + \frac{\partial \mathcal{A}(\mathbf{r}, t)}{\partial \kappa(t)}. \tag{113}$$

While the changes of the coefficients of the first fundamental form (Eq. 10) are universally accepted as stretching measures, the theories differ in respect of the quantities used as bending measures [34]. In analogy to the stretching measures, the changes of the coefficients of the second fundamental form

[2] Relations that describe the connections between two physical quantities. Examples of constitutive equations are the Hooke's law, the Ohm's law, the thermal conductivity, and the Navier's equations.

(Eq. 45) have been used as bending measures and the internal energy $A(\mathbf{r}, t)$ assumes the following aspect [15, 1, 10, 11, 8]

$$\mu A(\mathbf{r}, t) = \varepsilon(t)^T C_{11} \varepsilon(t) + \kappa(t)^T C_{22} \kappa(t). \tag{114}$$

According to Grinspun et al., this formulation permits at most developable reference configurations [24]. Motivated by the fact that many surfaces are not developable, they propose to choose the changes of the mean curvatures as the bending measures. Very impressive effects of crease and crumple have been achieved [47, 5]. However, it is not clear in their proposal how they distinguish for example the bending effects in a sailcloth from the ones in a silk textile material, without resorting to the Gaussian curvature. Although both fabrics possess the same "creasing" or "buckling" behavior, in the former, the folds look stiffer – high resistance to the Gaussian curvature, and in the latter, wrinkles are much softer – low resistance to the Gaussian curvature. Besides, Thomaszewski et al. pointed in [44] that the stretching and the bending measures are not explicitly available in the proposal of Grinspun et al.

Aiming at an accurate and consistent way of representing bending energy, which may reliably reproduce characteristic behavior of different textile types, Thomaszewski et al. propose to model cloth as Kirchhoff-Love thin shell [44]. Since Kirchhof-Love theory is only valid for cases where the thickness approaches zero and the deformation is not too large, they further suggest to separately handle the strains that are not rotationally invariant under large deformations by using rotational strain formulation. Instead, we propose in this paper to employ the theory of a Cosserat surface for cloth modeling. A Cosserat surface is a surface embedded in R^3 to which an out-of-plane vector $\mathbf{d}$, called a *director*, is assigned to every point. The theory of a Cosserat surface is exact, complete, and fully consistent with dynamical and thermodynamical principles of continuum mechanics. It was originally proposed by the Cosserats in 1909, rediscovered during the 50s for oriented bodies modeling [18] and, later, for shell modeling [23].

Applying the general Cosserat's shell theory to cloth modeling is not a novelty. Eischen et al. present in [17] a cloth model founded on a Cosserat surface, after the publication of a series of three papers by Simo et al, in which they demonstrate that, despite its awkward formulation, a classical shell theory is conducive to an efficient numerical implementation [40, 41, 42]. The key point for their finding is a new parametrization that avoids the terms such as the Christoffel symbols and the coefficients of the second fundamental form. The price that they pay is to adopt relations that do not explicitly associate shape quantities with the statical ones. The main contribution of our work is to demonstrate that modeling the cloth as an inextensible normal-director elastic Cosserat surface we may get an algebraic expression for $A(\mathbf{r}, t)$ with explicit relations between the geometrical and statical quantities, It contains the components of both Gaussian and mean curvatures, that is familiar to the graphics community. A wider range of deformation patterns have been, thus, achieved: we may not only control the smoothness of the wrinkles and folds, but also their quantity on a given surface area.

4 Cosserat Surface

In this section only formulas that we use in our work are transcribed in order to make clear our real contribution. We refer to the detailed explanation in [20] as further reading.

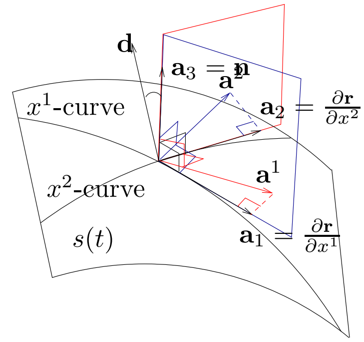

Figure 16: A moving trihedron.

Let $s(t) = \mathbf{r}(x^1, x^2, t)$ be a elastically deformable, smooth and non-intersecting surface at time t, where x^1- and x^2-curves are the coordinate curves lying on s. The x^i are identified as *convected coordinates* because any point on s has the same curvilinear coordinates in the reference state and in the deformed state[3]. Also, let $s(t)$ be referred to a fixed right-handed rectangular Cartesian coordinate system (x, y, z) defined by the transformation relations

$$x = x(x^1, x^2) \quad y = y(x^1, x^2) \quad z = z(x^1, x^2).$$

For the sake of conciseness, the Einstein notation (Section 2.4.7) is used in this section.

The first derivatives along the x^α-curves are

$$\mathbf{a}_\alpha(t) = \frac{\partial \mathbf{r}}{\partial x^\alpha}(t) = \mathbf{r}_{,\alpha}(t) \tag{115}$$

and the unit normal to $s(t)$ is given by

$$\mathbf{n}(t) = \mathbf{a}_3(t) = \frac{\frac{\partial \mathbf{r}}{\partial x^1}(t) \times \frac{\partial \mathbf{r}}{\partial x^2}(t)}{|\frac{\partial \mathbf{r}}{\partial x^1}(t) \times \frac{\partial \mathbf{r}}{\partial x^2}(t)|} = \frac{\mathbf{r}_{,\alpha}(t) \times \mathbf{r}_{,\beta}(t)}{|\mathbf{r}_{,\alpha}(t) \times \mathbf{r}_{,\beta}(t)|}. \tag{116}$$

These quantities are linearly independent and build the *basis vectors* of a *moving trihedron*, of which $\mathbf{a}_1$ and $\mathbf{a}_2$ lie in the tangent plane normal to $\mathbf{n}$ (Figure 17). If $\mathbf{a}^i$ denote the *reciprocal basis vectors* of s , we have

$$\mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0 \qquad \mathbf{a}_3 \cdot \mathbf{a}_3 = 1 \qquad \mathbf{a}^3 = \mathbf{a}_3 = 1 \qquad \mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 > 0.$$

Thus, the *metric tensor* of the coordinate system (x^1, x^2, x^3), when evaluated on $x^3 = 0$ at time t (i.e. in the deformed configuration), is given by

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{117}$$

Let $a^{\alpha\beta}$ denote the components of its contravariant metric tensor and b_α^β the coefficients of the formulae of Weingarten. In addition, let $a_{\alpha\beta}$, $b_{\alpha\beta}$ be the first of s and its second fundameSntal forms, and $a = a_{11}a_{22} - a_{12}a_{21}$ the determinant of its covariant metric tensor in any configuration $s(t)$. In particular, we designate their initial values in the reference configuration $s(t_0) = \mathcal{S}$ by $A_{\alpha\beta}$, $B_{\alpha\beta}$ and A, as well as the initial basis vectors by $\mathbf{A}_i$ and the initial position vector by $\mathbf{R}$.

Assigning to every point of $s(t)$ a deformable director $\mathbf{d} = \mathbf{d}(x^1, x^2, t)$, not necessarily along the normal to $s(t)$ and having the property that it remains invariant in magnitude under rigid motions of the surface, we have a *Cosserat surface* (Figure 16). The assigned director is intended to portray the "thickening"about the surface s and its component along the unit normal to s can be regarded as representing the "thickness" of s. In initial configuration we may specify the initial director $\mathbf{D}$ to be directed along $\mathbf{A}_3$ and its magnitude as representing the initial thickness of $s(t_0)$.

According to Eq. 19 an element of area of the surface $s(t)$ may be expressed as

$$d\sigma = \|\mathbf{a}_1 \times \mathbf{a}_2\| \, dx^1 dx^2 = (\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3) dx^1 dx^2 = a^{\frac{1}{2}} dx^1 dx^2. \tag{118}$$

In addition, in accordance with the equation of mass conservation, if $\rho(t)$ denotes the mass density of $s(t)$, we have

$$m(s) = m(\mathcal{S}) = \int\int_{x^1, x^2} \rho(x^1, x^2, t) a^{\frac{1}{2}} dx^1 dx^2 = \int\int_{x^1, x^2} \rho(x^1, x^2, t_0) A^{\frac{1}{2}} dx^1 dx^2,$$

even though both ρ and a may depend on t. An immediate consequence is the relation of the elements of area in the deformed and undeformed configuration

$$\rho(x^1, x^2, t_0) = \rho(x^1, x^2, t) (\frac{a}{A})^{\frac{1}{2}} \tag{119}$$

4.1 Kinematics of a Cosserat Surface

The motion of a *Cosserat surface* is describable by the vector functions $\mathbf{r}$ and $\mathbf{d}$. Figure 17 illustrates the change of the shape $s(t^*)$ to the shape $s(t^{**})$ by displacing each point $\mathbf{r}$ and/or altering its director. If we assume that these functions are differentiable with respect to x^α and t as many times as required, we may define the velocity $\mathbf{v}$ of a point of $s(t)$ and the director velocity $\mathbf{w}$ at time t.

[3] The coordinate curves of $s(t)$ will not generally be lines of curvature of s.

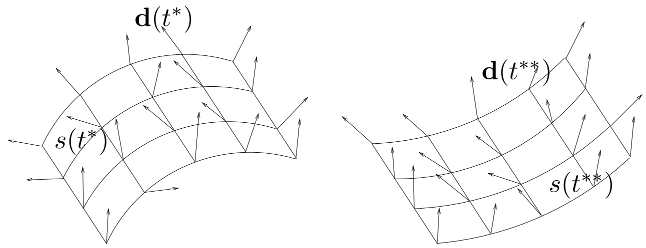

Figure 17: A moving Cosserat surface.

4.1.1 Material Derivatives

Since the coordinate curves on s are convected, the time rate of change of the basis vectors $\mathbf{a}_\alpha$ is given by

$$\dot{\mathbf{a}}_\alpha = \frac{\partial}{\partial t}(\frac{\partial \mathbf{r}}{\partial x^\alpha}) = \frac{\partial}{\partial x^\alpha}(\frac{\partial \mathbf{r}}{\partial t}).$$

They are also known as *material derivatives* of $\mathbf{a}_\alpha$. If we take the velocity $\mathbf{v}$ with respect to the basis $\mathbf{a}_i = \{\mathbf{a}_\alpha, \mathbf{a}_3\}$, namely

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t} = v^i \mathbf{a}_i = v_i \mathbf{a}^i,$$

and derive it with respect to the convected coordinates, we get

$$\begin{aligned} \dot{\mathbf{a}}_\alpha = \frac{\partial}{\partial x^\alpha}(\frac{\partial \mathbf{r}}{\partial t}) = \mathbf{v}_{,\alpha} &= v^i_{,\alpha}\mathbf{a}_i + v^i \mathbf{a}_{i,\alpha} = v^\beta_{,\alpha}\mathbf{a}_\beta + v^\beta \mathbf{a}_{\beta,\alpha} + v^3_{,\alpha}\mathbf{a}_3 + v^3 \mathbf{a}_{3,\alpha} \\ &= v_{i,\alpha}\mathbf{a}^i + v_i \mathbf{a}^i_{,\alpha} = v_{\beta,\alpha}\mathbf{a}^\beta + v_\beta \mathbf{a}^\beta_{,\alpha} + v_{3,\alpha}\mathbf{a}^3 + v_3 \mathbf{a}^3_{,\alpha} \\ \dot{\mathbf{a}}_3 = \mathbf{v}_{,3} &= v^i_{,3}\mathbf{a}_i + v^i \mathbf{a}_{i,3} = v_{i,3}\mathbf{a}^i + v_i \mathbf{a}^i_{,3}. \end{aligned}$$

Using Eq. 79 and Eq. 81, we may write

$$\begin{aligned} \dot{\mathbf{a}}_\alpha = \mathbf{v}_{,\alpha} &= v^\beta_{,\alpha}\mathbf{a}_\beta + v^\beta(\Gamma^\rho_{\beta\alpha}\mathbf{a}_\rho + b_{\beta\alpha}\mathbf{a}_3) + v^3_{,\alpha}\mathbf{a}_3 + v^3(-b^\beta_\alpha \mathbf{a}_\beta) \\ &= (v^\beta_{,\alpha} + v^\beta \Gamma^\rho_{\beta\alpha})\mathbf{a}_\beta + v^\beta b_{\beta\alpha}\mathbf{a}_3 + v^3_{,\alpha}\mathbf{a}_3 - v^3 b^\beta_\alpha \mathbf{a}_\beta \\ &= (v^\beta_{|\alpha} - v^3 b^\beta_\alpha)\mathbf{a}_\beta + (v^\beta b_{\beta\alpha} + v^3_{,\alpha})\mathbf{a}_3. \end{aligned}$$

And with use of Eq. 80 and Eq. 83, together with $\mathbf{a}^3 = \mathbf{a}_3$, we obtain an alternative formulation to the material derivatives

$$\begin{aligned} \dot{\mathbf{a}}_\alpha = \mathbf{v}_{,\alpha} &= v_{\beta,\alpha}\mathbf{a}^\beta + v_\beta(-\Gamma^\beta_{\rho\alpha}\mathbf{a}^\rho + b^\beta_\alpha \mathbf{a}_3) + v_{3,\alpha}\mathbf{a}^3 + v_3(-b_{\beta\alpha}\mathbf{a}^\beta) \\ &= (v_{\beta,\alpha} - v_\beta \Gamma^\beta_{\rho\alpha})\mathbf{a}^\beta + v_\beta b^\beta_\alpha \mathbf{a}^3 + v_{3,\alpha}\mathbf{a}^3 - v_3 b_{\beta\alpha}\mathbf{a}^\beta \\ &= (v_{\beta|\alpha} - v_3 b_{\beta\alpha})\mathbf{a}^\beta + (v_\beta b^\beta_\alpha + v_{3,\alpha})\mathbf{a}^3 \\ &= (v_{\beta|\alpha} - v_3 b_{\beta\alpha})\mathbf{a}^\beta + (v_\beta b^\beta_\alpha + v_{3,\alpha})\mathbf{a}_3. \end{aligned} \tag{120}$$

As $\mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0$, $\mathbf{a}^\alpha \cdot \mathbf{a}_3 = 0$ and $\mathbf{a}_3 \cdot \mathbf{a}_3 = 1$, we have $\dot{\mathbf{a}}_\alpha \cdot \mathbf{a}_3 + \mathbf{a}_\alpha \cdot \dot{\mathbf{a}}_3 = 0 \Longrightarrow \mathbf{a}_\alpha \cdot \dot{\mathbf{a}}_3 = -\dot{\mathbf{a}}_\alpha \cdot \mathbf{a}_3$. In consequence,

$$\begin{aligned} \dot{\mathbf{a}}_3 &= -\mathbf{a}^\alpha(\dot{\mathbf{a}}_\alpha \cdot \mathbf{a}_3) \\ &= -\mathbf{a}^\alpha(((v^\beta_{|\alpha} - v^3 b^\beta_\alpha)\mathbf{a}_\beta + (v^\beta b_{\beta\alpha} + v^3_{,\alpha})\mathbf{a}_3) \cdot \mathbf{a}_3) \\ &= -\mathbf{a}^\alpha(((v_{\beta|\alpha} - v_3 b_{\beta\alpha})\mathbf{a}^\beta + (v_\beta b^\beta_\alpha + v_{3,\alpha})\mathbf{a}_3) \cdot \mathbf{a}_3) \\ &= -(v^\beta b_{\beta\alpha} + v^3_{,\alpha})\mathbf{a}^\alpha = -(v_\beta b^\beta_\alpha + v_{3,\alpha})\mathbf{a}^\alpha. \end{aligned} \tag{121}$$

For convenience, let us summarize the gradient of a vector field $\mathbf{V}$ referred to the basis vectors $\mathbf{a}_i$ or to the reciprocal basis vectors $\mathbf{a}^i$

$$\mathbf{V}_{,\alpha} = \mathbf{V}_{|\alpha} = V_{i\alpha}\mathbf{a}^i = V^i_{,\alpha}\mathbf{a}_i, \tag{122}$$

where

$V_{i\alpha} = \mathbf{a}_i \cdot \mathbf{V}_{,\alpha}$ $\quad V^i_{,\alpha} = \mathbf{a}^i \cdot \mathbf{V}_{,\alpha}$
$V_{\lambda\alpha} = V_{\lambda|\alpha} - b_{\alpha\lambda}V_3$ $\quad V^\lambda_{,\alpha} = a^{\lambda\beta}V_{\beta\alpha} = V^\lambda_{|\alpha} - b^\lambda_\alpha V_3$
$V_{3\alpha} = V_{3,\alpha} + b^\lambda_\alpha V_\lambda$ $\quad V^3_{,\alpha} = V_{3,\alpha} = V^3_{,\alpha} + b_{\lambda\alpha}V^\lambda$.

In particular, we may express the material derivatives as a linear combination of the reciprocal basis vectors $\mathbf{a}^i$

$$\dot{\mathbf{a}}_i = c_{ki}\mathbf{a}^k \Longrightarrow c_{ki} = \mathbf{a}_k \cdot \dot{\mathbf{a}}_i.$$

The elements c_{ki} build a square matrix which may be expressed as the sum of symmetric η and antisymmetric parts ψ

$$C = \begin{bmatrix} \mathbf{a}^1 \cdot \dot{\mathbf{a}}_1 & \mathbf{a}^1 \cdot \dot{\mathbf{a}}_2 & \mathbf{a}^1 \cdot \dot{\mathbf{a}}_3 \\ \mathbf{a}^2 \cdot \dot{\mathbf{a}}_1 & \mathbf{a}^2 \cdot \dot{\mathbf{a}}_2 & \mathbf{a}^2 \cdot \dot{\mathbf{a}}_3 \\ \mathbf{a}^3 \cdot \dot{\mathbf{a}}_1 & \mathbf{a}^3 \cdot \dot{\mathbf{a}}_2 & \mathbf{a}^3 \cdot \dot{\mathbf{a}}_3 \end{bmatrix} = \frac{1}{2}(C + C^T) + \frac{1}{2}(C - C^T) = \eta + \psi.$$

The components $\eta_{\alpha\beta}$ are referred to as *surface deformation rate tensor* and $\psi_{\alpha\beta}$, *surface spin tensor*. The material derivatives may be expressed in function of them

$$\dot{\mathbf{a}}_i = (\eta_{ki} + \psi_{ki})\mathbf{a}^k. \tag{123}$$

Applying Eq. 120 and Eq. 121 we may write the components $\eta_{\alpha\beta}$ and $\psi_{\alpha\beta}$ in terms of the velocity and the differential geometry quantities

$$\begin{aligned}
2\eta_{\alpha\beta} &= 2\eta_{\beta\alpha} = \mathbf{a}_\alpha \cdot \dot{\mathbf{a}}_\beta + \mathbf{a}_\beta \cdot \dot{\mathbf{a}}_\alpha \\
&= \mathbf{a}_\alpha \cdot ((v_{\alpha|\beta} - v_3 b_{\alpha\beta})\mathbf{a}^\alpha + (v_\alpha b^\alpha_\beta + v_{3,\beta})\mathbf{a}_3) + \\
&\quad \mathbf{a}_\beta \cdot ((v_{\beta|\alpha} - v_3 b_{\beta\alpha})\mathbf{a}^\beta + (v_\beta b^\beta_\alpha + v_{3,\alpha})\mathbf{a}_3) \\
&= (v_{\alpha|\beta} - v_3 b_{\alpha\beta}) + (v_{\beta|\alpha} - v_3 b_{\beta\alpha}) = v_{\alpha|\beta} + v_{\beta|\alpha} - 2v_3 b_{\alpha\beta} \\
2\eta_{\alpha 3} &= 2\eta_{3\alpha} = \mathbf{a}_\alpha \cdot \dot{\mathbf{a}}_3 + \mathbf{a}_3 \cdot \dot{\mathbf{a}}_\alpha \\
&= = (v_\alpha b^\alpha_\beta + v_{3,\beta}) + (v_\alpha b^\alpha_\beta + v_{3,\beta}) = 0 \\
2\eta_{33} &= \mathbf{a}_3 \cdot \dot{\mathbf{a}}_3 + \mathbf{a}_3 \cdot \dot{\mathbf{a}}_3 = 0
\end{aligned} \tag{124}$$

$$\begin{aligned}
2\psi_{\alpha\beta} &= -2\psi_{\alpha\beta} = \mathbf{a}_\alpha \cdot \dot{\mathbf{a}}_\beta - \mathbf{a}_\beta \cdot \dot{\mathbf{a}}_\alpha = v_{\alpha|\beta} - v_{\beta|\alpha} \\
2\psi_{\alpha 3} &= -2\psi_{3\alpha} = \mathbf{a}_\alpha \cdot \dot{\mathbf{a}}_3 - \mathbf{a}_3 \cdot \dot{\mathbf{a}}_\alpha = -2(v_\beta b^\beta_\alpha + v_{3,\alpha}) \\
2\psi_{33} &= \mathbf{a}_3 \cdot \dot{\mathbf{a}}_3 - \mathbf{a}_3 \cdot \dot{\mathbf{a}}_3 = 0
\end{aligned} \tag{125}$$

4.1.2 Change Rate of Reciprocal Basis Vectors

It is also disable to obtain the time rate of change of the reciprocal basis vectors of a Cosserat surface when it is subjected to a velocity field.

Taking the derivative of Eq. 27

$$\begin{aligned}
&d(\mathbf{a}^{\alpha\lambda}\mathbf{a}_{\lambda\beta}) = 0 \\
&\Longrightarrow \dot{\mathbf{a}}^{\alpha\beta}\mathbf{a}_{\lambda\alpha} + \mathbf{a}^{\beta\upsilon}\dot{\mathbf{a}}_{\lambda\upsilon} = 0 \\
&\Longrightarrow \dot{\mathbf{a}}^{\alpha\beta}\mathbf{a}_{\lambda\alpha} = -\mathbf{a}^{\beta\upsilon}\dot{\mathbf{a}}_{\lambda\upsilon} \\
&\Longrightarrow \dot{\mathbf{a}}^{\alpha\beta} = -\mathbf{a}^{\lambda\alpha}\mathbf{a}^{\beta\upsilon}\dot{\mathbf{a}}_{\lambda\upsilon}
\end{aligned}$$

together with Eq. 21, Eq. 123, and Eq. 124, we obtain the desired result in terms of $\eta_{\alpha\beta}$ and $\psi_{\alpha\beta}$

$$\begin{aligned}
\dot{\mathbf{a}}^\alpha &= \dot{\mathbf{a}}^{\alpha\lambda}\mathbf{a}_\lambda + \mathbf{a}^{\alpha\lambda}\dot{\mathbf{a}}_\lambda \\
&= (-\mathbf{a}^{\alpha\beta}\mathbf{a}^{\lambda\upsilon}\dot{\mathbf{a}}_{\beta\upsilon})\mathbf{a}_\lambda + \mathbf{a}^{\alpha\lambda}((\eta_{\kappa\lambda} + \psi_{\kappa\lambda})\mathbf{a}^\kappa) \\
&= (-\mathbf{a}^{\alpha\beta}\mathbf{a}^{\lambda\upsilon}(2\eta_{\beta\upsilon}))\mathbf{a}_\lambda + \mathbf{a}^{\alpha\lambda}((\eta_{\kappa\lambda} + \psi_{\kappa\lambda})\mathbf{a}^\kappa) \\
&= \mathbf{a}^{\alpha\lambda}((\psi_{\kappa\lambda} - \eta_{\kappa\lambda})\mathbf{a}^\kappa) \\
\dot{\mathbf{a}}^3 &= \dot{\mathbf{a}}_3 = \psi_{\kappa 3}\mathbf{a}^\kappa.
\end{aligned} \tag{126}$$

4.1.3 Change Rate of Director Vector

Kinematical results in terms of the director displacement $\mathbf{d}$ are also of interest to characterize the deformation and motion of a Cosserat surface. If we refer $\mathbf{d}$ to the basis vectors $\mathbf{a}^i$

$$\mathbf{d} = d_i\mathbf{a}^i = d_\alpha\mathbf{a}^\alpha + d_3\mathbf{a}^3,$$

then from Eq. 126

$$
\begin{aligned}
\mathbf{w} = \dot{\mathbf{d}} &= \dot{d}_\alpha \mathbf{a}^\alpha + d_\alpha \dot{\mathbf{a}}^\alpha + \dot{d}_3 \mathbf{a}^3 + d_3 \dot{\mathbf{a}}^3 \\
&= \dot{d}_i \mathbf{a}^i + d_\alpha \mathbf{a}^{\alpha\lambda}((\psi_{\kappa\lambda} - \eta_{\kappa\lambda})\mathbf{a}^\kappa) + d_3 \psi_{\kappa 3} \mathbf{a}^\kappa \\
&= \dot{d}_i \mathbf{a}^i + d_\alpha \mathbf{a}^{\alpha\lambda}((\psi_{\kappa\lambda} - \eta_{\kappa\lambda})\mathbf{a}^\kappa) + d^3 \psi_{\kappa 3} \mathbf{a}^\kappa \\
&= \dot{d}_i \mathbf{a}^i + d^\lambda(-\eta_{\kappa\lambda})\mathbf{a}^\kappa + d^i \psi_{\kappa i} \mathbf{a}^\kappa. \\
&= (\dot{d}_k + d^i(\psi_{ki} - \eta_{ki}))\mathbf{a}^k. \qquad (127)
\end{aligned}
$$

If we further define

$$\vec{\eta}_\lambda = -\eta_{\kappa\lambda}\mathbf{a}^\kappa,$$

we may get a more compact formulation

$$\mathbf{w} = \dot{\mathbf{d}} = \dot{d}_i \mathbf{a}^i + d^\lambda \vec{\eta}_\lambda + d^i \psi_{\kappa i} \mathbf{a}^\kappa. \qquad (128)$$

According to Eq 122, the gradient of $\mathbf{d}$ is

$$\mathbf{d}_{,\alpha} = \lambda_{i\alpha}\mathbf{a}^i = \lambda_{\beta\alpha}\mathbf{a}^\beta + \lambda_{3\alpha}\mathbf{a}^3 = (d_{\beta|\alpha} - b_{\beta\alpha})\mathbf{a}^\beta + (d_{3,\alpha} - b^\beta_\alpha d_\beta)\mathbf{a}^3. \qquad (129)$$

Also, from Eq. 127 we may have the gradient of the director velocity $\mathbf{w}$ or the time rate of change of the director derivative with respect to the convected coordinates

$$
\begin{aligned}
\mathbf{w}_{,\alpha} = \dot{\mathbf{d}}_{,\alpha} &= (\dot{d}_k \mathbf{a}^k)_{,\alpha} + (d^i \psi_{ki} \mathbf{a}^k)_{,\alpha} - (d^i \eta_{ki} \mathbf{a}^k)_{,\alpha} \\
&= (\dot{d}_k \mathbf{a}^k)_{,\alpha} + (d^i \psi_{ki} \mathbf{a}^k)_{,\alpha} - (d^i \vec{\eta}_i)_{,\alpha} \\
&= (\dot{d}_k \mathbf{a}^k)_{,\alpha} + \lambda^i_{,\alpha} \psi_{ki} \mathbf{a}^k - \lambda^\beta_{,\alpha} \vec{\eta}_\beta \\
&= \dot{\lambda}_{i\alpha} + \lambda^i_{,\alpha} \psi_{ki} \mathbf{a}^k - \lambda^\beta_{,\alpha} \vec{\eta}_\beta, \qquad (130)
\end{aligned}
$$

where

$$\lambda^\beta_{,\alpha} = a^{\beta\gamma}\lambda_{\gamma\alpha} \qquad \lambda^3_{,\alpha} = \lambda_{3,\alpha} \qquad (131)$$

4.1.4 Change Rate of Area Element

To compute the time rate of change of an element area given by Eq. 19, we use Eq. 23 and Eq. 124 to derive

$$
\begin{aligned}
\dot{a} = \frac{\partial(a_{11}a_{22} - a_{12}a_{21})}{\partial t} &= a_{22}\dot{a}_{11} + a_{11}\dot{a}_{22} - a_{12}\dot{a}_{21} - a_{21}\dot{a}_{12} \\
&= a(a^{11})\dot{a}_{11} + a(a^{22})\dot{a}_{22} + a(a^{21})\dot{a}_{21} + a(a^{12})\dot{a}_{12} \\
&= a(a^{11}(2v_{1|1} - 2b_{11}v_3) + a^{22}(2v_{2|2} - 2b_{22}v_3)) + \\
&\quad a(a^{21}(v_{2|1} + v_{1|2} - 2b_{12}v_3) + a^{12}(v_{1|2} + v_{2|1} - 2b_{21}v_3)) \\
&= 2a((a^{11}v_1 + a^{12}v_2)_{|1} - (a^{11}b_{11} + a^{12}b_{21})v_3) + \\
&\quad 2a((a^{21}v_1 + a^{22}v_2)_{|2} - (a^{21}b_{12} + a^{22}b_{22})v_3) \\
&= 2a((v^1_{|1} - b^1_1 v_3) + (v^2_{|2} - b^2_2 v_3)) \equiv 2a(v^\alpha_\alpha - b^\alpha_\alpha v_3).
\end{aligned}
$$

Defining

$$\eta^\alpha_\alpha = \frac{1}{2}a^{-1}\dot{a},$$

we may rewrite the integrand and the domain of integration of Eq. 19 as the element of area

$$d\sigma = a^{\frac{1}{2}}dx^1dx^2.$$

Then,

$$
\begin{aligned}
\dot{d\sigma} &= \frac{\partial(a^{\frac{1}{2}})}{\partial t}dx^1dx^2 \\
&= \frac{1}{2}a^{-\frac{1}{2}}\dot{a}dx^1dx^2 = \frac{1}{2}a^{-1}a^{\frac{1}{2}}\dot{a}dx^1dx^2 \\
&= \eta^\alpha_\alpha d\sigma = (v^\alpha_\alpha - b^\alpha_\alpha v_3)d\sigma \qquad (132)
\end{aligned}
$$

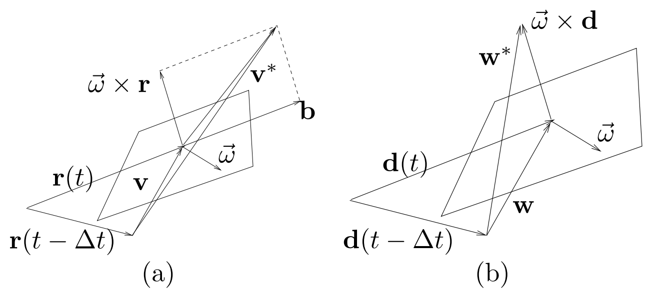

Figure 18: Superposed rigid body motions: (a) position velocity and (b) director velocity.

4.1.5 Superposed Rigid Body Motions

It is interesting to know whether the above-mentioned kinematical quantities remain invariant under superposed rigid motions $\mathbf{b}$ and $\vec{\omega}$, representable by W, an orthogonal tensor-valued function of t or an affine transformation matrix. The vector $\mathbf{b}$ may be interpreted as a uniform rigid body translational velocity and $\vec{\omega}$ is a uniform rigid body angular velocity at time t. Under such motions, the position $\mathbf{r}$ and the director $\mathbf{d}$ at $\mathbf{r}$ are displaced to the position $\mathbf{r}^*$ and the director $\mathbf{d}^*$ at $\mathbf{r}^*$, such that

$$\begin{aligned}\mathbf{r}^* &= \mathbf{r}^*(x^1,x^2,t) = \mathbf{r}^*(x^1,x^2,t_0) + W(\mathbf{r}(x^1,x^2,t) - \mathbf{r}(x^1,x^2,t_0)) = \\ &\quad \mathbf{R}^*(x^1,x^2) + W(\mathbf{r}(x^1,x^2,t) - \mathbf{R}(x^1,x^2)) \\ \mathbf{d}^* &= \mathbf{d}^*(x^1,x^2,t) = W(\mathbf{d}(x^1,x^2,t)). \end{aligned} \tag{133}$$

In this case, the magnitude of the relative displacement $|\mathbf{r}-\mathbf{R}|$ remains unaltered

$$|\mathbf{r}^* - \mathbf{R}^*| = |\mathbf{r} - \mathbf{R}|.$$

Hence, the element of area of the surface s and its mass density ρ remains unaltered. In addition, all kinematical quantities related with the material coordinates, such as $a_{\alpha\beta}$ and $b_{\alpha\beta}$, are preserved

$$\mathbf{a}^*_\alpha = \mathbf{a}_\alpha \quad \mathbf{a}^*_3 = \mathbf{a}_3 \quad \mathbf{d}^*_{,\alpha} = \mathbf{d}_{,\alpha} \quad a^*_{\alpha\beta} = a_{\alpha\beta} \quad b^*_{\alpha\beta} = b_{\alpha\beta}$$

The superposed velocity and the superposed director velocity become (Fig. 18)

$$\begin{aligned}\dot{\mathbf{r}^*} = \mathbf{v}^* &= \mathbf{v} + [\mathbf{b} + \vec{\omega} \times \mathbf{r}], \\ \dot{\mathbf{d}^*} = \mathbf{w}^* &= \mathbf{w} + \vec{\omega} \times \mathbf{d}\end{aligned}$$

The superposed velocity gradient at time t is then

$$\dot{\mathbf{a}^*}_\alpha = \mathbf{v}^*_{,\alpha} = \mathbf{v}_{,\alpha} + [\mathbf{b} + \vec{\omega} \times \mathbf{r}]_{,\alpha} = \mathbf{v}_{,\alpha} + \vec{\omega} \times \mathbf{r}_\alpha = \dot{\mathbf{a}}_\alpha + \vec{\omega} \times \mathbf{a}_\alpha.$$

Since the cross product can alternatively be defined in terms of the ϵ-system (Eq. 2.2) and the basis vectors and their reciprocal ones satisfy Eq. 22, we may define

$$\begin{aligned}\omega^m &= \vec{\omega} \cdot \mathbf{a}^m \\ \Omega_{k\alpha} &= \epsilon_{k\alpha m}\omega^m = e_{k\alpha m}\sqrt{a}\omega^m\end{aligned}$$

and rewrite the expression as

$$\dot{\mathbf{a}^*}_\alpha = \dot{\mathbf{a}}_\alpha + \vec{\omega} \times \mathbf{a}_\alpha = \dot{\mathbf{a}}_\alpha - (e_{k\alpha m}\sqrt{a}\omega^m)\mathbf{a}^k = \dot{\mathbf{a}}_\alpha - \Omega_{k\alpha}\mathbf{a}^k. \tag{134}$$

If we replace $\mathbf{a}_i$ in Eq. 124 and Eq. 125 by $\dot{\mathbf{a}}_i + \vec{\omega} \times \mathbf{a}_i$, we get

$$\eta^*_{ki} = \eta_{ki} \qquad \psi^*_{ki} = \psi_{ki} - \Omega_{ki} \tag{135}$$

From Eq. 128 we deduce the superposed director velocity

$$\begin{aligned}\dot{\mathbf{d}}^* = \mathbf{w}^* &= \mathbf{w} + \vec{\omega} \times \mathbf{d} \\ &= (\dot{d}^*_k + d^{i*}(\psi^*_{ki} - \eta^*_{ki}))\mathbf{a}^k \\ &= (\dot{d}_k + d^i(\psi_{ki} - \Omega_{ki} - \eta^*_{ki}))\mathbf{a}^k \\ &= (\dot{d}_k + d^i(\psi_{ki} - \eta^*_{ki}))\mathbf{a}^k - d^i\Omega_{ki}\mathbf{a}^k \\ &= \mathbf{w} + d^i\Omega_{ik}\mathbf{a}^k\end{aligned} \tag{136}$$

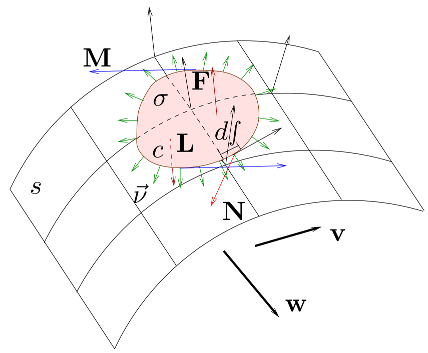

Figure 19: Forces and couples acting on s.

and its gradient follows directly from Eq. 130

$$\begin{aligned}
\dot{\mathbf{d}}^*_{,\alpha} = \mathbf{w}^*_{,\alpha} &= (\dot{d}^*_k \mathbf{a}^k)_{,\alpha} + \lambda^{i*}_{,\alpha}\psi^*_{ki}\mathbf{a}^k - \lambda^{\beta*}_{,\alpha}\vec{\eta}^*_\beta \\
&= (\dot{d}_k \mathbf{a}^k)_{,\alpha} + \lambda^{i}_{,\alpha}(\psi_{ki} - \Omega_{ki})\mathbf{a}^k - \lambda^{\beta}_{,\alpha}\vec{\eta}_\beta \\
&= \mathbf{w}_{,\alpha} - \lambda^{i}_{,\alpha}\Omega_{ki}\mathbf{a}^k \\
&= \mathbf{w}_{,\alpha} + \lambda^{i}_{,\alpha}\Omega_{ik}\mathbf{a}^k
\end{aligned} \tag{137}$$

4.1.6 Alternative Kinematic Measures

Besides the kinematic quantities presented, which involve mainly $a_{\alpha\beta}$, $\lambda_{\alpha\beta}$, d_i, and their derivatives, it is often convenient to use the following measures

1. **Membrane strains** ($\varepsilon_{\alpha\beta}$)

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}) \tag{138}$$

2. **Bending strains** ($\kappa_{\beta i}$)

$$\begin{aligned}
\kappa_{\alpha\beta} &= \lambda_{\alpha\beta} - \Lambda_{\alpha\beta} \\
\kappa_{3\alpha} &= \lambda_{3\alpha} - \Lambda_{3\alpha},
\end{aligned} \tag{139}$$

where $\Lambda_{i\alpha} = \mathbf{A}_i\mathbf{D}_{,\alpha}$ are the values of $\lambda_{i\alpha}$ in the initial configuration.

3. **Director or transverse shear strains** (δ_i)

$$\delta_i = d_i - D_i. \tag{140}$$

4.2 Dynamics of a Cosserat Surface

These strains are results of the internal forces due to the action of the proper surface or the external loading forces. We proceed to characterize the basic field equations for a Cosserat surface.

Let σ be the area of a Cosserat surface s bounded by a closed curve c. Let $c = c(x^1(\int), x^2(\int))$ be the parametric equations of the curve with $\int$ as the arc parameter; and let $\vec{\tau}$ represent the unit tangent vector to c. Then

$$\vec{\tau} = \frac{\partial \mathbf{r}}{\partial \int} = \frac{\partial \mathbf{r}}{\partial x^1}\frac{dx^1}{d\int} + \frac{\partial \mathbf{r}}{\partial x^2}\frac{dx^2}{d\int} = \mathbf{a}_1\tau^1 + \mathbf{a}_2\tau^2 \equiv \tau^\alpha \mathbf{a}_\alpha$$

and the unit normal vector $\vec{\nu}$ to c lying on the surface is

$$\vec{\nu} = \vec{\tau} \times \mathbf{a}_3 = \nu_\alpha \mathbf{a}^\alpha = \nu^\alpha \mathbf{a}_\alpha = \epsilon_{\alpha\beta 3}\tau^\beta \mathbf{a}_\alpha = \sqrt{a}\tau^1\mathbf{a}_2 - \sqrt{a}\tau^2\mathbf{a}_1, \tag{141}$$

where ϵ_{ijk} has been introduced in Section. 2.2. Conversely, τ can be expressed as

$$\vec{\tau} = \mathbf{a}_3 \times \vec{\nu} = \mathbf{a}_3 \times \nu_\alpha \mathbf{a}^\alpha = \frac{1}{\sqrt{a}}e^{\alpha\beta 3}\nu_\alpha \mathbf{a}_\beta = \frac{1}{\sqrt{a}}\nu_1\mathbf{a}_2 - \frac{1}{\sqrt{a}}\nu_2\mathbf{a}_1. \tag{142}$$

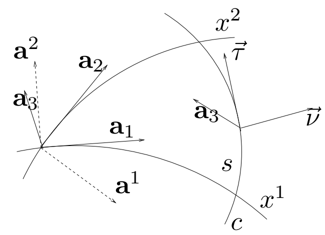

Figure 20: A curvilinear triangle.

If for all arbitrary velocity fields $\mathbf{v}$, there is a three-dimensional force field $\mathbf{N} = N^i\mathbf{a}_i$ defined for points of c, such that the scalar $\mathbf{N} \cdot \mathbf{v}$ represents a rate of work per unit length $d\int$ of c, then $\mathbf{N}$ is a *contact force* (or *curve force*) vector measured per unit length of c that acts across c. Similarly, if $\mathbf{M} = M^i\mathbf{a}_i$ is a three-dimensional vector field and if, for all arbitrary director velocity fields $\mathbf{w}$, the scalar $\mathbf{M} \cdot \mathbf{w}$ represents a rate of work per unit length of c, then $\mathbf{M}$ is a *contact director* (or a *curve director*) couple measured per unit length of c (Figure 19). In an analogous way, we may define $\mathbf{F} = F^i\mathbf{a}_i$ and $\mathbf{L} = L^i\mathbf{a}_i$ as three-dimensional vector fields per unit mass for points on σ. If $\mathbf{F} \cdot \mathbf{v}$ is a rate of work per unit mass, then $\mathbf{F}$ is an *assigned force* per unit mass of s, and if $\mathbf{L} \cdot \mathbf{w}$ is a rate of work per unit mass, then it is called an *assigned director couple* per unit mass of s at time t.

If ρ is the mass density at time t per unit area of s and U is the internal energy per unit mass, then the equation of conservation of energy may be written as

$$\frac{\partial(\int_\sigma[\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \rho U]d\sigma)}{\partial t} = \int_\sigma \rho[\vartheta + \mathbf{F}\cdot\mathbf{v} + \bar{\mathbf{L}}\cdot\mathbf{w}]d\sigma + \int_c [\mathbf{N}\cdot\mathbf{v} + \mathbf{M}\cdot\mathbf{w}]dc - \int_c hdc,$$

$$\int_\sigma [\rho\mathbf{v}\cdot\dot{\mathbf{v}} + \rho\dot{U}]d\sigma + \int_\sigma \{\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + U\}[\dot{\rho} + \rho(v^\alpha_{|\alpha} - b^\alpha_\alpha v_3)]d\sigma =$$
$$\int_\sigma \rho[\vartheta + \mathbf{F}\cdot\mathbf{v} + \bar{\mathbf{L}}\cdot\mathbf{w}]d\sigma + \int_c [\mathbf{N}\cdot\mathbf{v} + \mathbf{M}\cdot\mathbf{w}]dc - \int_c hdc,$$
$$\int_\sigma \rho\dot{U}d\sigma + \int_\sigma \{\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + U\}[\dot{\rho} + \rho(v^\alpha_{|\alpha} - b^\alpha_\alpha v_3)]d\sigma =$$
$$\int_\sigma \rho[\vartheta + (\mathbf{F} - \dot{\mathbf{v}})\cdot\mathbf{v} + \bar{\mathbf{L}}\cdot\mathbf{w}]d\sigma + \int_c [\mathbf{N}\cdot\mathbf{v} + \mathbf{M}\cdot\mathbf{w}]dc - \int_c hdc, \quad (143)$$

where $\bar{\mathbf{L}}$ is the difference of the assigned director force per unit mass $\mathbf{L}$ and the inertia terms due to the director displacement $\mathbf{d}$, ϑ is the specific heat supply (or heat absorption) by radiation per unit mass per unit time, h is the scalar heat flux across c, by conduction, per unit lenght per unit time.

In this section we present five basic sets of fied equations for a Cosserat surface: conservation of mass (Eq. 146), linear momentum principle (Eq. 150), director momentum principle (Eq. 157), moment of momentum (Eq. 160), and the balance of energy (Eq. 168).

4.2.1 Elementary Curvilinear Triangle

According to Eq. 143 the resultant contact force and the resultant contact director couple exerted on σ at time t are defined by the line integrals over the boundary c. It is, therefore, convenient to relate the element of arc length of each coordinate curve, ds_α, to the element of arc length of c. ds.

Starting with the fact

$$ds_1 = \frac{dx^1}{\sqrt{a_{11}}} \qquad ds_2 = \frac{dx^2}{\sqrt{a_{22}}}.$$

If we adopt the convention that if one proceeds along a closed curve, the bounding area is always kept to the left, the unit tangent vectors to the coordinate curves are

$$\vec{\tau}^{(1)} = \frac{\mathbf{a}_1}{\sqrt{a_{11}}} \qquad \vec{\tau}^{(2)} = \frac{\mathbf{a}_2}{\sqrt{a_{22}}}$$

and the outward unit normal vectors to them are

$$\vec{\nu}^{(1)} = -\frac{\mathbf{a}^2}{\sqrt{a^{22}}} \qquad \vec{\nu}^{(2)} = -\frac{\mathbf{a}^1}{\sqrt{a^{11}}}. \quad (144)$$

Using Eq. 142, we may write

$$
\begin{aligned}
-\frac{\mathbf{a}_1}{\sqrt{a_{11}}}ds_1 + \frac{\mathbf{a}_2}{\sqrt{a_{22}}}ds_2 &= e^{\alpha\beta 3}\nu_\alpha \mathbf{a}_\beta ds \\
&= [\epsilon^{113}\frac{1}{\sqrt{a}}\mathbf{a}_1\nu_1 + \epsilon^{123}\frac{1}{\sqrt{a}}\mathbf{a}_2\nu_1 + \epsilon^{213}\frac{1}{\sqrt{a}}\mathbf{a}_1\nu_2 + \epsilon^{223}\frac{1}{\sqrt{a}}\mathbf{a}_2\nu_2]ds \\
&= [\frac{1}{\sqrt{a}}\nu_1\mathbf{a}_2 - \frac{1}{\sqrt{a}}\nu_2\mathbf{a}_1]ds
\end{aligned}
$$

Matching the terms on the both sides of the equal sign, we have

$$
\begin{aligned}
\frac{1}{\sqrt{a_{11}}}ds_1 = \frac{1}{\sqrt{a}}\nu_2 ds \Longrightarrow ds_1 = \frac{\sqrt{a_{11}}}{\sqrt{a}}\nu_2 ds = \sqrt{a^{22}}\nu_2 ds \\
\frac{1}{\sqrt{a_{22}}}ds_2 = \frac{1}{\sqrt{a}}\nu_1 ds \Longrightarrow ds_2 = \frac{\sqrt{a_{22}}}{\sqrt{a}}\nu_1 ds = \sqrt{a^{11}}\nu_1 ds.
\end{aligned} \tag{145}
$$

4.2.2 Contact Force Field

Differentiating the terms on the right-hand side of Eq. 143 and supposing that the quantities ρ, U, ϑ, h, $\mathbf{F} = \dot{\mathbf{v}}$, $\bar{\mathbf{L}}$, $\mathbf{N}$, $\mathbf{M}$ and $\mathbf{w}$ remain unchanged under superposed uniform rigid body translational velocities $\mathbf{b}$, that is after replacing $\mathbf{v}$ by $(\mathbf{v}+\mathbf{b})$ in Eq 143, and subtracting Eq. 143 from it, we get

$$
\begin{aligned}
\mathbf{b}\cdot\{\int_\sigma(\rho\dot{\mathbf{v}} - \rho\mathbf{F})d\sigma - \int_c \mathbf{N}dc + \int_\sigma \mathbf{v}[\dot{\rho} + \rho(v^\alpha_{|\alpha} - b^\alpha_\alpha v_3)]d\sigma\} + \\
\frac{1}{2}(\mathbf{b}\cdot\mathbf{b})\int_\sigma[\dot{\rho} + \rho(v^\alpha_{|\alpha} - b^\alpha_\alpha v_3)]d\sigma = 0
\end{aligned}
$$

But, since the law of the conversation of mass, together with Eq. 132, allows us to state

$$
\int_\sigma \frac{\partial(\rho d\sigma)}{\partial t} = \int_\sigma \mathbf{v}[\dot{\rho} + \rho(v^\alpha_{|\alpha} - b^\alpha_\alpha v_3)]d\sigma\} = 0, \tag{146}
$$

we have

$$
\int_\sigma(\rho\dot{\mathbf{v}} - \rho\mathbf{F})d\sigma - \int_c \mathbf{N}dc = 0. \tag{147}
$$

Let $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ be the physical force vectors acting on te sides of the coordinate curves with the outward unit normal vectors $\frac{\mathbf{a}^1}{\sqrt{a^{11}}}$ and $\frac{\mathbf{a}^2}{\sqrt{a^{22}}}$, so that for the outward unit vector given by Eq. 144 they are $-\mathbf{n}^{(1)}$ and $-\mathbf{n}^{(2)}$. Let $\mathbf{N}$ denote the physical force over any curve c with the outward unit normal $\vec{\nu}$. In the limit, the curvilinear triangle shown in Fig. 20 approaches its boundary curve and Eq. 147 becomes

$$
\mathbf{N}dc = \mathbf{N}ds - \mathbf{n}^{(2)}ds_1 - \mathbf{n}^{(1)}ds_2.
$$

Applying Eq. 145, we have

$$
\mathbf{N} = \mathbf{n}^{(2)}\sqrt{a^{22}}\nu_2 + \mathbf{n}^{(1)}\sqrt{a^{11}}\nu_1 \equiv \mathbf{N} = (\mathbf{n}^\alpha\sqrt{a^{\alpha\alpha}})\nu_\alpha = \mathbf{N}^\alpha\nu_\alpha, \text{ with } \mathbf{N}^\alpha = \mathbf{n}^\alpha\sqrt{a^{\alpha\alpha}},
$$

and expressing the contravariant surface vector $\mathbf{N}^\alpha$ as a linear combination of the basis vectors

$$
\mathbf{N}^\alpha = N^{\gamma\alpha}\mathbf{a}_\gamma + N^{3\alpha}\mathbf{a}_3, \tag{148}
$$

we may write

$$
\mathbf{N} = (N^{\gamma\alpha}\mathbf{a}_\gamma + N^{3\alpha}\mathbf{a}_3)\nu_\alpha = (N^{\gamma\alpha}\nu_\alpha)\mathbf{a}_\gamma + (N^{3\alpha}\nu_\alpha)\mathbf{a}_3. \tag{149}
$$

$N^{\gamma\alpha}$ and $N^{3\alpha}$ are surface tensors under transformation of the convected coordinates.

Substituting Eq. 148 into Eq. 147, making the usual smoothness assumptions and transforming the line integral into a surface integral by Stokes' theorem, it follows that

$$
\int_\sigma[\rho(\dot{\mathbf{v}} - \mathbf{F}) - \mathbf{N}^\alpha_{|\alpha}]d\sigma = 0
$$

and, in consequence, we have the first basic field equation of motion for the Cosserat surface

$$
\mathbf{N}^\alpha_{|\alpha} + \rho\mathbf{F} = \rho\dot{\mathbf{v}}. \tag{150}
$$

Taking the scalar product with $\mathbf{a}^\beta$

$$\mathbf{a}^\beta \cdot \mathbf{N}^\alpha_{|\alpha} + \rho \mathbf{a}^\beta(\mathbf{F} - \dot{\mathbf{v}}) = 0$$

and applying the derivative rule

$$(\mathbf{a}^\beta \cdot \mathbf{N}^\alpha)_{|\alpha} = \mathbf{a}^\beta \cdot \mathbf{N}^\alpha_{|\alpha} + \mathbf{a}^\beta_{|\alpha} \cdot \mathbf{N}^\alpha,$$

we have

$$(\mathbf{a}^\beta \cdot \mathbf{N}^\alpha)_{|\alpha} - \mathbf{N}^\alpha \cdot \mathbf{a}^\beta_{|\alpha} + \rho(\mathbf{F} - \dot{\mathbf{v}}) \cdot \mathbf{a}^\beta = 0.$$

Similarly, with $\mathbf{a}_3$ we get

$$(\mathbf{a}_3 \cdot \mathbf{N}^\alpha)_{|\alpha} - \mathbf{N}^\alpha \cdot \mathbf{a}_{3|\alpha} + \rho(\mathbf{F} - \dot{\mathbf{v}}) \cdot \mathbf{a}_3 = 0.$$

Further, letting c^α be the component of acceleration $\dot{\mathbf{v}}$ and using Eq. 80 and Eq. 81, we reach the component form of Eq. 150

$$\begin{aligned} N^{\beta\alpha}_{|\alpha} - b^\beta_\alpha N^{3\alpha} + \rho F^\beta &= \rho c^\beta \\ N^{3\alpha}_{|\alpha} - b_{\alpha\beta} N^{\beta\alpha} + \rho F^3 &= \rho c^3 \end{aligned} \tag{151}$$

4.2.3 Contact Director Couple

Using Eq. 146 and Eq. 150, Eq. 143 assumes the following aspect

$$\begin{aligned} \int_\sigma \rho \dot{U} d\sigma + \int_\sigma \{\frac{1}{2}\mathbf{v} \cdot \mathbf{v} + U\}[\dot{\rho} + \rho(v^\alpha_{|\alpha} - b^\alpha_\alpha v_3)] d\sigma = \int_\sigma \rho \dot{U} d\sigma = \\ \int_\sigma [\rho\vartheta + \rho(\mathbf{F} - \dot{\mathbf{v}}) \cdot \mathbf{v} + \rho \bar{\mathbf{L}} \cdot \mathbf{w}] d\sigma + \int_c [\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \mathbf{w}] dc - \int_c h dc = \\ \int_\sigma \rho[\vartheta + \bar{\mathbf{L}} \cdot \mathbf{w}] d\sigma + \int_\sigma \rho(\mathbf{F} - \dot{\mathbf{v}}) \cdot \mathbf{v} d\sigma + \rho + \int_c [\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \mathbf{w}] dc - \int_c h dc = \\ \int_\sigma \rho[\vartheta + \bar{\mathbf{L}} \cdot \mathbf{w}] d\sigma - \int_\sigma \mathbf{N}^\alpha_{|\alpha} \cdot \mathbf{v} d\sigma + \rho + \int_c [\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \mathbf{w}] dc - \int_c h dc \end{aligned}$$

On the other hand, we have from the Stokes' theorem

$$\int_c \mathbf{N} \cdot \mathbf{v} dc = \int_\sigma (\mathbf{N} \cdot \mathbf{v})_{|\alpha} d\sigma = \int_\sigma (\mathbf{N}^\alpha_{|\alpha} \cdot \mathbf{v} + \mathbf{N}^\alpha \cdot \mathbf{v}_{,\alpha}) d\sigma.$$

Putting them together, we reduce the energy balance (Eq. 143) to

$$\begin{aligned} \int_\sigma \rho \dot{U} d\sigma = \\ \int_\sigma \rho[\vartheta + \bar{\mathbf{L}} \cdot \mathbf{w}] d\sigma - \int_\sigma \mathbf{N}^\alpha_{|\alpha} \cdot \mathbf{v} d\sigma + \rho + \int_\sigma (\mathbf{N}^\alpha_{|\alpha} \cdot \mathbf{v} + \mathbf{N}^\alpha \cdot \mathbf{v}_{,\alpha}) d\sigma + \\ \int_c \mathbf{M} \cdot \mathbf{w} dc - \int_c h dc = \\ \int_\sigma [\rho(\vartheta + \bar{\mathbf{L}} \cdot \mathbf{w}) + \mathbf{N}^\alpha \cdot \mathbf{v}_{,\alpha}] d\sigma + \int_c \mathbf{M} \cdot \mathbf{w} dc - \int_c h dc. \end{aligned} \tag{152}$$

If we replace $\mathbf{w}$ by $(\mathbf{w} + \omega \times \mathbf{d})$ and $\mathbf{v}$ by $(\mathbf{v} + \omega \times \mathbf{r})$, then by subtraction we can deduce

$$\int_\sigma (\mathbf{a}_\alpha \times \mathbf{N}^\alpha + \rho \mathbf{d} \times \bar{\mathbf{L}}) d\sigma + \int_c (\mathbf{d} \times \mathbf{M}) dc = 0. \tag{153}$$

Let $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ be the physical director couple vectors acting on the sides of the x^1- and x^2-curves whose outward unit normal vectors are $\frac{\mathbf{a}^1}{\sqrt{a^{11}}}$ and $\frac{\mathbf{a}^2}{\sqrt{a^{22}}}$. Let $\mathbf{M}$ denote the physical contact director couple acting on any curve c with the outward unit normal $\vec{\nu}$. By an argument similar to that which led to Eq. 148, the curvilinear triangle shown in Fig. 20 approaches its boundary curve in the limit and the application of the director momentum to an elementary curvilinear triangle yields

$$\mathbf{M} = \mathbf{M}^1\nu_1 + \mathbf{M}^2\nu_2 \qquad \mathbf{M}^\alpha = \mathbf{m}^{(\alpha)}\sqrt{a^{\alpha\alpha}}. \tag{154}$$

$\mathbf{M}^\alpha$ transforms as a contravariant surface vector. In addition, the line integral in Eq. 153 may be transformed into a surface integral by Stokes' theorem

$$\int_\sigma [\rho(\vartheta - \dot{U} + \bar{\mathbf{L}}\cdot\mathbf{w}) + \mathbf{N}^\alpha\cdot\mathbf{v}_{,\alpha}]d\sigma + \int_\sigma (\mathbf{M}^\alpha_{|\alpha}\cdot\mathbf{w} + \mathbf{M}^\alpha\cdot\mathbf{w}_{,\alpha})d\sigma - \int_c hdc = 0. \tag{155}$$

If we set

$$\mathbf{M}^\alpha = M^{\gamma\alpha}\mathbf{a}_\gamma + M^{3\alpha}\mathbf{a}_3, \tag{156}$$

then the components $M^{i\alpha}$ are surface tensors.

If we admit the existence of a vector field $\mathbf{l}$, an *intrisic director couple* per unit area of s, we also have the following conservation law – the director momemtum principle

$$\mathbf{l} = \rho\bar{\mathbf{L}} + \mathbf{M}^\alpha_{|\alpha}, \tag{157}$$

Taking the scalar product with $\mathbf{a}^\beta$ or $\mathbf{a}_3$ and applying Eq. 80 and Eq. 81, we may obtain its component form in a similar way that we get the component form of Eq. 150

$$\begin{aligned} M^{\beta\alpha}_{|\alpha} - b^\beta_\alpha M^{3\alpha} + \rho\bar{L}^\beta &= l^\beta \\ M^{3\alpha}_{|\alpha} - b_{\alpha\beta}M^{\beta\alpha} + \rho\bar{L}^3 &= l^3. \end{aligned} \tag{158}$$

Then by combining it with Eq 155, we have

$$\int_\sigma [\rho\vartheta - \rho\dot{U} + \mathbf{l}\cdot\mathbf{w}) + \mathbf{N}^\alpha\cdot\mathbf{v}_{,\alpha} + \mathbf{M}^\alpha\cdot\mathbf{w}_{,\alpha}]d\sigma - \int_c hdc = 0, \tag{159}$$

and with Eq. 153, we get

$$\mathbf{a}_\alpha \times \mathbf{N}^\alpha + \mathbf{d}\times\mathbf{l} + \mathbf{d}_{,\alpha}\times\mathbf{M}^\alpha = 0. \tag{160}$$

Using the permutation symbol (Section 2.2), we may rewrite it in component form

$$\begin{aligned} e_{jim}(\delta^j_\alpha N^{i\alpha}) + e_{jim}(d^j l^i) + e_{jim}(\lambda^j_{,\alpha}M^{i\alpha}) &= \\ e_{jim}[\delta^j_\alpha N^{i\alpha} + d^j l^i + \lambda^j_{,\alpha}M^{i\alpha}] &= \\ N^{3\alpha} + (l^3 d^\alpha - l^\alpha d^3) + M^{3\gamma}\lambda^\alpha_{,\gamma} - M^{\alpha\gamma}\lambda^3_{,\gamma} &= 0 \end{aligned} \tag{161}$$

4.2.4 Flux of Heat

Let $h^{(1)}$ and $h^{(2)}$ be the flux of heat across the coordinate curves whose outward unit normal vectors are $\frac{\mathbf{a}}{\sqrt{a_{11}}}$ and $\frac{\mathbf{a}}{\sqrt{a_{22}}}$. Let q^α the contravariant components of the heat flux vector across any curve c with the outward unit normal $\vec{\nu}$. By an argument similar to that which led to Eq. 148, the curvilinear triangle shown in Fig. 20 approaches its boundary curve in the limit and Eq. 159, becomes

$$hdc = q^\alpha\nu_\alpha ds - h^{(2)}ds_1 - h^{(1)}ds_2 = 0 \Longrightarrow q^\alpha = h^{(\alpha)}\sqrt{a^{\alpha\alpha}}. \tag{162}$$

Transforming the line integral in Eq. 159 into a surface integral, we finally get for an arbitraty surface area

$$\begin{aligned} \int_\sigma [\rho\vartheta - q^\alpha_{|\alpha} - \rho\dot{U} + \mathbf{l}\cdot\mathbf{w} + \mathbf{N}^\alpha\cdot\mathbf{v}_{,\alpha} + \mathbf{M}^\alpha\cdot\mathbf{w}_{,\alpha}]d\sigma = 0 \\ \rho\vartheta - q^\alpha_{|\alpha} - \rho\dot{U} + \mathbf{l}\cdot\mathbf{w} + \mathbf{N}^\alpha\cdot\mathbf{v}_{,\alpha} + \mathbf{M}^\alpha\cdot\mathbf{w}_{,\alpha} = 0. \end{aligned} \tag{163}$$

4.2.5 Superposed Rigid Body Motion

In Section 4.1.5 we have seen that the element of area and the mass density remain unaltered under superposed rigid body motions. In this section, it is of our interest to verify how the forces **N**, **M**, **F** and **L** behave under these motions. Then, we will see how the basic field equations for a Cosserat surface should be changed.

In Section 4.2.2 and Section 4.2.3 we have seen that **N** and **M** are defined over a curve c on the surface with the outward unit normal vector $\vec{\nu}$. Under superposed rigid body motions the orientation of the surface may change in 3D space, so the outward normal vector of the same material point may change and becomes $\vec{\nu}^*$. Nevertheless, because the transformations are distance preserving, we expect that the forces $\mathbf{N}^*$ and $\mathbf{M}^*$ satisfy the following conditions:

- $\|\mathbf{N}^*\| = \|\mathbf{N}\|$ and $\|\mathbf{M}^*\| = \|\mathbf{M}\|$;
- to have the same orientation relative to $\vec{\nu}^*$ as they have relative to $\vec{\nu}$.

The force fields **F**, **L** and **l** are defined throughout a region of s. From Eq. 150 and Eq. 157 we may deduce that they transform in the same way as $\vec{\nu}$ under the superposed distance preserving transformations.

Once under superposed rigid body motion the shape of the Cosserat surface keeps invariant, we should again deduce Eq. 160 and Eq. 163

$$\mathbf{a}^*_\alpha \times \mathbf{N}^{\alpha *} + \mathbf{d}^* \times \mathbf{l}^* + \mathbf{d}^*_{,\alpha} \times \mathbf{M}^{\alpha *} = \mathbf{a}_\alpha \times \mathbf{N}^\alpha + \mathbf{d} \times \mathbf{l} + \mathbf{d}_{,\alpha} \times \mathbf{M}^\alpha = 0.$$

$$\begin{aligned} \rho^*\vartheta^* - q^{\alpha *}_{|\alpha} - \rho^*\dot{U}^* + \mathbf{l}^* \cdot \mathbf{w}^* + \mathbf{N}^{\alpha *} \cdot \mathbf{v}^*_{,\alpha} + \mathbf{M}^{\alpha *} \cdot \mathbf{w}^*_{,\alpha} \\ \rho\vartheta - q^{\alpha}_{|\alpha} - \rho\dot{U} + \mathbf{l} \cdot \mathbf{w}^* + \mathbf{N}^{\alpha} \cdot \mathbf{v}^*_{,\alpha} + \mathbf{M}^{\alpha} \cdot \mathbf{w}^*_{,\alpha} = 0 \end{aligned}$$

From Eq. 123, Eq. 134, Eq. 136, Eq. 137 and the fact that $\psi_{ki} = \Omega_{ki}$ under superposed rigid body motions, we may rewrite Eq. 163 in an invariant form

$$\begin{aligned} \rho\vartheta - q^{\alpha}_{|\alpha} - \rho\dot{U} + \mathbf{l} \cdot \mathbf{w}^* + \mathbf{N}^{\alpha} \cdot \mathbf{v}^*_{,\alpha} + \mathbf{M}^{\alpha} \cdot \mathbf{w}^*_{,\alpha} = \\ \rho\vartheta - q^{\alpha}_{|\alpha} - \rho\dot{U} + \mathbf{l} \cdot ((\dot{d}_k + d^i(\psi_{ki} - \eta_{ki}))\mathbf{a}^k - d^i\Omega_{ki}\mathbf{a}^k) + \\ \mathbf{N}^\alpha \cdot ((\eta_{k\alpha} + \psi_{k\alpha}) - \Omega_{k\alpha})\mathbf{a}^k) + \\ \mathbf{M}^\alpha((\dot{d}_k\mathbf{a}^k)_{,\alpha} + \lambda^i_{,\alpha}(\psi_{ki} - \Omega_{ki})\mathbf{a}^k - \lambda^\beta_{,\alpha}\vec{\eta}_\beta) = \\ \rho\vartheta - q^{\alpha}_{|\alpha} - \rho\dot{U} + \mathbf{l} \cdot ((\dot{d}_k - d^\alpha\eta_{k\alpha})\mathbf{a}^k + \mathbf{N}^\alpha\eta_{k\alpha}\mathbf{a}^k + \\ \mathbf{M}^\alpha \cdot ((\dot{d}_k\mathbf{a}^k)_{,\alpha} - \lambda^\beta_{,\alpha}\vec{\eta}_\beta) = \\ \rho\vartheta - q^{\alpha}_{|\alpha} - \rho\dot{U} + (\mathbf{N}^\alpha \cdot \eta_{k\alpha} - \mathbf{l} \cdot d^\alpha\eta_{k\alpha} - \mathbf{M}^\gamma\lambda^\beta_{,\gamma}\eta_{k\gamma})\mathbf{a}^k + \\ \mathbf{l} \cdot \dot{d}_k\mathbf{a}^k + \mathbf{M}^\alpha \cdot (\dot{d}_k\mathbf{a}^k)_{,\alpha} = \\ \rho\vartheta - q^{\alpha}_{|\alpha} - \rho\dot{U} + (\mathbf{N}^\alpha - \mathbf{l} \cdot d^\alpha - \mathbf{M}^\gamma\lambda^\beta_{,\gamma})\eta_{k\alpha}\mathbf{a}^k + \\ \mathbf{l} \cdot \dot{d}_k\mathbf{a}^k + \mathbf{M}^\alpha \cdot (\dot{d}_k\mathbf{a}^k)_{,\alpha} = \\ \rho\vartheta - q^{\alpha}_{|\alpha} - \rho\dot{U} + \mathbf{N}'^{\alpha}\eta_{k\alpha}\mathbf{a}^k + \mathbf{l} \cdot \dot{d}_k\mathbf{a}^k + \mathbf{M}^\alpha \cdot (\dot{d}_k\mathbf{a}^k)_{,\alpha}, \end{aligned}$$

where

$$\mathbf{N}'^{\alpha} = \mathbf{N}^\alpha - \mathbf{l} \cdot d^\alpha - \mathbf{M}^\gamma\lambda^\beta_{,\gamma}.$$

In component forms, we have

$$\rho\vartheta - q^{\alpha}_{|\alpha} - \rho\dot{U} + N'^{\beta\alpha}\eta_{\alpha\beta} + l^k\dot{d}_k + \mathbf{M}^{k\alpha}\dot{\lambda}_{k\alpha} \tag{164}$$

with the symmetric components

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\beta\alpha} - m^\alpha d^\beta - M^{\alpha\gamma}\lambda^\beta_{,\gamma} \tag{165}$$

4.3 Thermodynamic

Eq. 164, which involves the specific internal energy U, can be written in terms of an alternative thermodynamic potential, namely the specific *Helmholtz free energy*. For this purpose, we introduce a Helmholtz funcion per unit mass

$$A = U - TS, \tag{166}$$

where S is the *specific entropy* per unit mass and the positive value T is the temperature. The surface integral over an area σ of s

$$\int_\sigma \rho S d\sigma$$

is the *entropy* of σ and the *production of entropy* per unit time in σ at time t postulated by the following inequality

$$\int_\sigma \rho \dot{S} d\sigma - \int_\sigma \rho \frac{\vartheta}{T} d\sigma - \int_c \frac{h}{T} dc \geq 0. \tag{167}$$

The second term in Eq. 167 is the entropy due to radiation entering σ and the third term is the flux of entropy due to conduction entering σ through the boundary c.

Replacing U in Eq. 164 by $A - TS$, we have

$$\begin{aligned} \rho\vartheta - q^\alpha_{|\alpha} - \rho \dot{\overline{A - TS}} + N^{'\beta\alpha}\eta_{\alpha\beta} + l^k \dot{d}_k + \mathbf{M}^{k\alpha}\dot{\lambda}_{k\alpha} = \\ \rho\vartheta - q^\alpha_{|\alpha} - \rho\dot{A} - \rho(\dot{T}S + T\dot{S}) + N^{'\beta\alpha}\eta_{\alpha\beta} + l^k \dot{d}_k + \mathbf{M}^{k\alpha}\dot{\lambda}_{k\alpha} = 0. \end{aligned} \tag{168}$$

Replacing h in Eq. 167 by Eq. 162 and applying Stokes' theorem to the line integral, we obtain

$$\begin{aligned} \int_\sigma \rho\dot{S} d\sigma - \int_\sigma \rho\frac{\vartheta}{T} d\sigma - \int_c \frac{q^\alpha}{T} dc = \\ \int_\sigma \rho\dot{S} d\sigma - \int_\sigma \rho\frac{\vartheta}{T} d\sigma - \int_\sigma (\frac{q^\alpha_{|\alpha}}{T} - \frac{q^\alpha T_{,\alpha}}{T^2}) d\sigma = \\ \int_\sigma \rho T\dot{S} d\sigma - \int_\sigma \rho\vartheta d\sigma - \int_\sigma (q^\alpha_{|\alpha} - \frac{q^\alpha T_{,\alpha}}{T}) d\sigma = \\ \int_\sigma (\rho T\dot{S} d\sigma - \rho\vartheta - (q^\alpha_{|\alpha} - \frac{q^\alpha T_{,\alpha}}{T})) d\sigma \geq 0, \end{aligned}$$

which leads to the following inequality

$$\rho T\dot{S} d\sigma - \rho\vartheta - (q^\alpha_{|\alpha} - \frac{q^\alpha T_{,\alpha}}{T}) \geq 0 \tag{169}$$

5 Elastic Cosserat Surface

We define an elastic Cosserat surface as one for which the following constitutive equations hold for all time t

$$\begin{aligned} A &= A(T, \varepsilon_{\alpha\beta}, \kappa_{\beta i}, \delta_i, \Lambda_{\beta i}, D_i), \\ S &= S(T, \varepsilon_{\alpha\beta}, \kappa_{\beta i}, \delta_i, \Lambda_{\beta i}, D_i), \\ q^\alpha &= q^\alpha(T, T_{,\gamma}, \varepsilon_{\gamma\beta}, \kappa_{\beta i}, \delta_i, \Lambda_{\beta i}, D_i), \\ h &= h(T, \varepsilon_{\alpha\beta}, \kappa_{\beta i}, \delta_i, \Lambda_{\beta i}, D_i, \nu_\alpha), \\ M^i &= M^i(T, \varepsilon_{\alpha\beta}, \kappa_{\beta i}, \delta_i, \Lambda_{\beta i}, D_i, \nu_\alpha), \\ N^{'\alpha\beta} &= N^{'\alpha\beta}(T, \varepsilon_{\gamma\delta}, \kappa_{\gamma i}, \delta_i, \Lambda_{\gamma i}, D_i), \\ m^i &= m^i(T, \varepsilon_{\gamma\delta}, \kappa_{\gamma j}, \delta_j, \Lambda_{\gamma j}, D_j), \\ M^{i\alpha} &= M^{i\alpha}(T, \varepsilon_{\gamma\delta}, \kappa_{\gamma j}, \delta_j, \Lambda_{\gamma j}, D_j), \end{aligned} \tag{170}$$

where the strains $\varepsilon_{\alpha\beta}$, $\kappa_{\beta i}$ and δ_i have been defined in Section 4.1.6.

Combining Eq.169 and Eq. 168, we reach the following energy equation

$$-\rho\dot{T}S - \rho\dot{A} + N^{'\beta\alpha}\eta_{\alpha\beta} + l^k\dot{d}_k + \mathbf{M}^{k\alpha}\dot{\lambda}_{k\alpha} - \frac{q^\alpha T_{,\alpha}}{T} \geq 0. \tag{171}$$

From the constitutive equations,

$$\dot{A} = \frac{\partial A}{\partial T} dT + \frac{\partial A}{\partial \varepsilon_{\alpha\beta}} d\varepsilon_{\alpha\beta} + \frac{\partial A}{\partial \delta_i} d\delta_i + \frac{\partial A}{\partial \kappa_{i\alpha}} d\kappa_{i\alpha}.$$

Then, by plugging it into Eq. 171, the expression becomes

$$-\rho(S + \frac{\partial A}{\partial T})\dot{T} + (N^{'\beta\alpha} - \rho\frac{\partial A}{\partial \varepsilon_{\alpha\beta}})\eta_{\alpha\beta} + (l^k - \rho\frac{\partial A}{\partial \delta_i})\dot{d}_k + (\mathbf{M}^{k\alpha} - \rho\frac{\partial A}{\partial \kappa_{i\alpha}})\dot{\lambda}_{k\alpha} - \frac{q^\alpha T_{,\alpha}}{T} \geq 0.$$

Green et al. state in [23] that, if the temperature distribution varies homogeneously with time, then

$$S = -\frac{\partial A}{\partial T} \qquad N^{'\alpha\beta} = \rho\frac{\partial A}{\partial \varepsilon_{\beta\alpha}} \qquad l^i = \rho\frac{\partial A}{\partial \delta_i} \qquad M^{i\alpha} = \rho\frac{\partial A}{\partial \kappa_{i\alpha}} \qquad -q^\alpha T_{,\alpha} \geq 0$$

When an elastic surface is initially *isotropic with a center of geometry* and assuming that A is a polynomial function with arguments indicated in Eq. 170 and A does not depend explicitly on $\Lambda_{i\alpha}$ and D_i, then we may write A as a function of the *joint invariants* of these arguments. For convenience, Green et al. introduce in [23] the notations

$$e^\alpha_\beta = A^{\alpha\gamma}\varepsilon_{\gamma\beta}$$

$$\kappa^\alpha_{,\beta} = A^{\alpha\gamma}\kappa_{\gamma\beta} \qquad \kappa^{,\beta}_\alpha = A^{\beta\gamma}\kappa_{\alpha\gamma} \qquad \kappa^{;\alpha}_3 = A^{\alpha\gamma}\kappa_{3\gamma} \qquad \delta^\alpha = A^{\alpha\gamma}\delta_\gamma$$

and the 2×2 marices

$$I = e^\alpha_\beta \qquad J = \kappa^\alpha_{,\beta} \qquad u = \kappa^{;\alpha}_3 \qquad v = \delta^\alpha$$
$$K = uu^T \qquad P = uv^T \qquad P^T = vu^T \qquad Q = vv^T. \tag{172}$$

Then, A may be expressed as a function of T, $\mathbf{d}$ and the following twenty-four joint invariants from the traces of matrices

$$\begin{array}{r} trace(I) \quad trace(J) \quad trace(K) \quad trace(P) \quad trace(Q) \\ trace(I^2) \quad trace(J^2) \quad trace(IJ) \quad trace(IJ^T) \\ trace(IK) \quad trace(IP) \quad trace(IQ) \\ trace(JK) \quad trace(JQ) \quad trace(JP) \quad trace(JP^T) \\ trace(IJJ^T) \\ trace(IJK) \quad trace(IJQ) \quad trace(IJP) \quad trace(IJP^T) \\ trace(JJ^TK) \quad trace(JJ^TQ) \quad trace(JJ^TP). \end{array}$$

5.1 Infinitesimal Deformations

Green et al. also show in [23] that if we consider that the material point displacements and the director displacements, as well as their material coordinate and time derivatives, all kinematical quantities and all contact forces, remain small of the order infinitesimal, Eq. 168 assumes the following aspect

$$\rho_0\vartheta - Q^\alpha_{|\alpha} - \rho_0\dot{A} - \rho(\dot{T}S + T\dot{S}) + N^{'\beta\alpha}\dot{\varepsilon}_{\alpha\beta} + l^i\dot{\delta}_i + \mathbf{M}^{k\alpha}\dot{\kappa}_{k\alpha} = 0. \tag{173}$$

and the constitutive equations become

$$h = \nu^\alpha_0 Q_\alpha \qquad N^{'\alpha\beta} = \rho_0\frac{\partial A}{\partial \varepsilon_{\beta\alpha}} \qquad l^i = \rho_0\frac{\partial A}{\partial \delta_i} \qquad M^{i\alpha} = \rho_0\frac{\partial A}{\partial \kappa_{i\alpha}}, \tag{174}$$

where ρ_0 is the initial mass density, Q_α are the components of the heat flux per unit length (in the undeformed surface) per unit time, and ν^α_0 are the components of the unit outward normal to the x^α-curves on s.

If the surface is initially homogeneous, free from curve and director forces, and in the state of rest at a constant temperature T_0 and entropy S_0, then to the order of approximation considered, it is sufficient to express $\rho_0 A$ as a quadratic function of $\varepsilon_{\alpha\beta}$, $\kappa_{i\alpha}$, δ_i and T, where T is now the temperature difference from T_0. Thus, if S_0

$$\begin{aligned} \rho_0 A \quad = \quad & C^{\alpha\beta\gamma\delta}_1\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + C^{\alpha\beta\gamma\delta}_2\kappa_{\alpha\beta}\kappa_{\gamma\delta} + C^{\alpha\beta\gamma\delta}_3\varepsilon_{\alpha\beta}\kappa_{\gamma\delta} + \\ & C^{\alpha\beta\gamma}_1\kappa_{3\alpha}\kappa_{\beta\gamma} + C^{\alpha\beta\gamma}_2\varepsilon_{\alpha\beta}\delta_\gamma + C^{\alpha\beta\gamma}_3\varepsilon_{\alpha\beta}\kappa_{3\gamma} + C^{\alpha\beta\gamma}_4\delta_\alpha\kappa_{\beta\gamma} + \\ & C^{\alpha\beta}_1\delta_\alpha\delta_\beta + C^{\alpha\beta}_2\kappa_{3\alpha}\kappa_{3\beta} + C^{\alpha\beta}_3\delta_\alpha\kappa_{3\beta} + \\ & C^{\alpha\beta}_4\varepsilon_{\alpha\beta}\delta_3 + C^{\alpha\beta}_5\kappa_{\alpha\beta}\delta_3 + C^{'\alpha\beta}_4\varepsilon_{\alpha\beta}T + C^{'\alpha\beta}_5\kappa_{3\alpha\beta}T + \\ & C^\alpha_1\delta_\alpha\delta_3 + C^\alpha_2\kappa_{3\alpha}\delta_3 + C^{'\alpha}_1\delta_\alpha T + C^{'\alpha}_2\kappa_{3\alpha}T + \\ & C(\delta_3)^2 + C^{'}T^2 + C^{''}\delta_3 T, \end{aligned}$$

where the coefficients C, $C^{'}$, C^α_n, $C^{'\alpha}_n$, $C^{\alpha\beta}_n$, $C^{\alpha\beta\gamma}_n$, and $C^{\alpha\beta\gamma\delta}_n$, $n \in 1,2,3,4,5$, are constants and some of them satisfy certain symmetry conditions

$$C^{\alpha\beta\gamma\delta}_1 = C^{\beta\alpha\gamma\delta}_1 = C^{\alpha\beta\delta\gamma}_1 = C^{\gamma\delta\alpha\beta}_1$$

$$C_2^{\alpha\beta\gamma\delta} = C_2^{\gamma\delta\alpha\beta} \qquad C_3^{\alpha\beta\gamma\delta} = C_3^{\beta\alpha\gamma\delta} \qquad C_2^{\alpha\beta\gamma} = C_2^{\beta\alpha\gamma} \tag{175}$$

If the surface possesses isotropy with a center of symmetry, a tensor basis is given by $A^{\alpha\beta}$ and all odd order coefficients must vanish

$$\begin{aligned} C_1^{\alpha\beta\gamma} = C_2^{\alpha\beta\gamma} = C_3^{\alpha\beta\gamma} = C_4^{\alpha\beta\gamma} = 0, \\ C_1^{\alpha} = C_2^{\alpha} = C_1^{'\alpha} = C_2^{'\alpha} = 0. \end{aligned} \tag{176}$$

In addition, the remaining coefficients must be homogeneous, linear functions of products of $A^{\alpha\beta}$, e.g.

$$\begin{aligned} C_1^{\alpha\beta\gamma\delta} &= \alpha_1 A^{\alpha\beta} A^{\gamma\delta} + \alpha_2 A^{\alpha\gamma} A^{\beta\delta} + \alpha_3 A^{\alpha\delta} A^{\beta\gamma} \\ C_1^{\beta\alpha\gamma\delta} &= \alpha_1' A^{\beta\alpha} A^{\gamma\delta} + \alpha_2' A^{\beta\gamma} A^{\alpha\delta} + \alpha_3' A^{\beta\delta} A^{\alpha\gamma}. \end{aligned}$$

Because of the symmetry conditions stated in Eq. 10, Eq. 175, Eq. 176, we have $\alpha_2 = \alpha_3$. Similar arguments can be applied to other coefficients and the free energy $\rho_0 A$ may be reduced to the expression

$$\begin{aligned} \rho_0 A = & \frac{1}{2}[\alpha_1 A^{\alpha\beta} A^{\gamma\delta} + \alpha_2 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + \\ & \frac{1}{2}\alpha_3 A^{\alpha\beta}\delta_\alpha\delta_\beta + \frac{1}{2}\alpha_4(\delta_3)^2 + \alpha_4^{'}\delta_3 T + \frac{1}{2}\alpha_4^{''} T^2 + \\ & \frac{1}{2}[\alpha_5 A^{\alpha\beta} A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma} A^{\beta\delta} + \alpha_7 A^{\alpha\delta} A^{\beta\gamma}]\kappa_{\alpha\beta}\kappa_{\gamma\delta} + \\ & \frac{1}{2}\alpha_8 A^{\alpha\beta}\kappa_{3\alpha}\kappa_{3\beta} + \alpha_9 A^{\alpha\beta}\varepsilon_{\alpha\beta}\delta_3 + \alpha_9^{'} A^{\alpha\beta}\varepsilon_{\alpha\beta} T + \\ & [\alpha_{10} A^{\alpha\beta} A^{\gamma\delta} + \alpha_{11}(A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\varepsilon_{\alpha\beta}\kappa_{\gamma\delta} + \\ & \alpha_{12} A^{\alpha\beta}\kappa_{\alpha\beta}\delta_3 + \alpha_{12}^{'} A^{\alpha\beta}\kappa_{\alpha\beta} T + \alpha_{13} A^{\alpha\beta}\delta_\alpha\kappa_{3\beta}. \end{aligned} \tag{177}$$

and the components of the curve, the intrinsic and the director forces (Eq. 174) assume the following form

$$\begin{aligned} N^{'\alpha\beta} = & [\alpha_1 A^{\alpha\beta} A^{\gamma\delta} + \alpha_2 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\beta} A^{\delta\gamma})]\varepsilon_{\gamma\delta} + \\ & [\alpha_{10} A^{\alpha\beta} A^{\gamma\delta} + \alpha_{11}(A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\kappa_{\gamma\delta} + \\ & \alpha_9 A^{\alpha\beta}\delta_3 + \alpha_9^{'} A^{\alpha\beta} T, \end{aligned} \tag{178}$$

$$\begin{aligned} l^\alpha &= \alpha_3 A^{\alpha\gamma}\delta_\gamma + \alpha_{13} A^{\alpha\gamma}\kappa_{3\gamma} \\ l^3 &= \alpha_4\delta_3 + \alpha_4^{'} T + \alpha_9 A^{\alpha\beta}\varepsilon_{\alpha\beta} + \alpha_{12} A^{\alpha\beta}\kappa_{\alpha\beta} \end{aligned} \tag{179}$$

$$\begin{aligned} M^{\alpha\beta} = & [\alpha_5 A^{\alpha\beta} A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma} A^{\beta\delta} + \alpha_7 A^{\alpha\delta} A^{\beta\gamma}]\kappa_{\gamma\delta} + \\ & [\alpha_{10} A^{\alpha\beta} A^{\gamma\delta} + \alpha_{11}(A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\varepsilon_{\gamma\delta} \\ & \alpha_{12} A^{\alpha\beta}\delta_3 + \alpha_{12}^{'} A^{\alpha\beta} T \\ M^{3\alpha} = & \alpha_8 A^{\alpha\gamma}\kappa_{3\gamma} + \alpha_{13} A^{\alpha\beta} T, \end{aligned} \tag{180}$$

where the coefficients α_n and $\alpha_n^{'}$, $n \in \{1, 2, ..., 13\}$, are constants.

5.2 Cosserat Surface with Undeformed Director

Green et al. further developed in [23] a set of basic force equations for the cases in which directors coincide with the (unit) normal vectors at any time t, that is $\mathbf{d}(t) = \mathbf{a}_3(t) = \mathbf{n}(t)$, they call it an *inextensible normal-director Cosserat surface*. In this case

$$d_\alpha = 0 \qquad d_3 = 1 \qquad D_\alpha = 0 \qquad D_3 = 1$$

From Eq. 129 it follows immediately

$$\lambda_{\alpha\beta} = -b_{\alpha\beta} \qquad \lambda_{3\alpha} = 0 \qquad \Lambda_{\alpha\beta} = -B_{\alpha\beta} \qquad \Lambda_{3\alpha} = 0,$$

and by replacing them in Eq. 139 we have

$$\kappa_{\alpha\beta} = -(b_{\alpha\beta} - B_{\alpha\beta}) \qquad \kappa_{3\beta} = 0. \tag{181}$$

Furthermore, from Eq. 121 we deduce

$$\mathbf{w} = \dot{\mathbf{d}} = \dot{\mathbf{a}}_3 = -(v_\beta b^\beta_\alpha + v_{3,\alpha})\mathbf{a}^\alpha. \tag{182}$$

The expression for material derivatives $\dot{\mathbf{a}}_i$ given by Eq. 120 and Eq. 121 remain, however, the same. The five basic sets of equations (Eq. 146, Eq. 150, Eq. 157, Eq. 161 and Eq. 168) may be simplified with use of Eq. 181, Eq. 182 and Eq 131.

Eq. 161 becomes

$$N^{3\alpha} = l^\alpha d^3 + M^{3\gamma}\lambda^\alpha_{,\gamma} = l^\alpha + M^{3\gamma}a^{\alpha\beta}\lambda_{\beta\gamma} = l^\alpha + M^{3\gamma}a^{\alpha\beta}b_{\beta\gamma} = l^\alpha + M^{3\gamma}b^\alpha_\gamma, \tag{183}$$

while $N^{\beta\alpha}$ that appears in Eq. 165 may be expressed as follows

$$\begin{aligned} N^{\beta\alpha} &= N'^{\beta\alpha} + l^\alpha d^\beta + M^{\alpha\gamma}\lambda^\beta_{,\gamma} = N'^{\beta\alpha} + M^{\alpha\gamma}\lambda^\beta_{,\gamma} = \\ &= N'^{\beta\alpha} - M^{\alpha\gamma}(a^{\beta\delta}b_{\delta\gamma}) = N'^{\beta\alpha} - M^{\alpha\gamma}b^\beta_\gamma. \end{aligned} \tag{184}$$

And, once $\dot{d}_K = 0$ and $\dot{\lambda}_{\beta\alpha} = \dot{b}_{\beta\alpha} = \dot{\kappa}_{\beta\alpha}$, Eq. 168, the equation of *balance of energy*, assumes the form

$$\rho\vartheta - q^\alpha_{|\alpha} - \rho\dot{A} - \rho(\dot{T}S + T\dot{S}) + N'^{\beta\alpha}\eta_{\alpha\beta} + \mathbf{M}^{\beta\alpha}\dot{\kappa}_{\beta\alpha} = 0. \tag{185}$$

If we combine Eq. 183 with Eq. 158, we have the modified expression for the *moment of momentum principle*

$$N^{3\beta} = M^{\beta\alpha}_{|\alpha} + \rho\bar{L}^\beta. \tag{186}$$

Putting it together with Eq. 151, we get an expression for the *linear momentum principle*

$$\begin{aligned} N^{\beta\alpha}_{|\alpha} - b^\beta_\gamma M^{\gamma\alpha}_{|\alpha} + \rho F^\beta &= \rho(c^\beta + b^\beta_\gamma \bar{L}^\gamma) \\ M^{\beta\alpha}_{|\alpha\beta} + b_{\alpha\beta}N^{\beta\alpha}_{|\alpha} + \rho F^3 &= \rho c^3 - (\rho\bar{L}^\gamma)_{|\beta}, \end{aligned} \tag{187}$$

and plugging it into Eq. 157, we obtain the modified component form for the *director momentum principle*

$$\begin{aligned} N^{3\alpha} - M^{3\gamma}b^\alpha_\gamma &= l^\alpha \\ M^{3\alpha}_{|\alpha} + b_{\alpha\beta}M^{\beta\alpha} + \rho\bar{L}^3 &= l^3. \end{aligned} \tag{188}$$

Finally, the free energy $\rho_0 A$ assumes the form

$$A = A(T, \varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, B_{\alpha\beta})$$

and the constitutive equations are reduced to

$$N'^{\alpha\beta} = \rho\frac{\partial A}{\partial\varepsilon_{\alpha\beta}} \qquad M^{\alpha\beta} = \rho\frac{\partial A}{\partial\kappa_{\alpha\beta}} \tag{189}$$

When the surface is initially isotropic with a center of gemetry, A may be expressed as a function of T and the joint invariants

$$\begin{array}{r} trace(I) \quad trace(J) \quad trace(L) \\ trace(I^2) \quad trace(J^2) \quad trace(L^2) \\ trace(IJ) \quad trace(IL) \quad trace(JL) \\ trace(IJL), \end{array} \tag{190}$$

where I and J are defined in Eq. 172 and $L = A^{\alpha\gamma}B_{\gamma\beta}$.

We remark that throughout this section and the following sections, all covariant differentiations are with respect to the metric tensor $A_{\alpha\beta}$ of the undeformed surface.

5.3 Discrete Cosserat Surfaces with Undeformed Director

The main task we should undertake is to make the analytic expressions developed for the Cosserat surface readily processable by a computer.

5.3.1 Membrane and Bending Strains

The simplicity of Eq 187, Eq. 138 and Eq. 139 does not reveal the complexity of its component equations. While Eq. 138 requires the first derivatives, Eq. 181 may involve the products of normal vectors and second derivatives. Fortunately, Section 2.5 already provides us a procedure for estimating the geometric quantities $a_{\alpha\beta}$ and $b_{\alpha\beta}$ for a reference system that takes into consideration the lengths of adjacent edges. Our problem is then reduced to the computation of the variations of reference systems $(\mathbf{new\_t}, \mathbf{new\_b}, \mathbf{new\_n})$ in time. For convenience, we denote the reference systems at time t by $(\mathbf{new\_t}(t), \mathbf{new\_b}(t), \mathbf{new\_n}(t))$.

We propose to track the variations of the edges $v_i v_{i,t}$ and $v_i v_{i,b}$ that have been chosen to build the reference systems in the rest state $(\mathbf{new\_t}(t_0), \mathbf{new\_b}(t_0), \mathbf{new\_n}(t_0))$. For each vertex v_i, the metric tensor at t_0 are the reference values that appear in Eq. 138

$$A_{11} = \mathbf{new\_t}(t_0)\cdot\mathbf{new\_t}(t_0) \qquad A_{12} = A_{21} = \mathbf{new\_t}(t_0)\cdot\mathbf{new\_b}(t_0) \qquad A_{22} = \mathbf{new\_b}(t_0)\cdot\mathbf{new\_b}(t_0),$$

while the coefficients of the curvature tensor with respect to the reference $(\mathbf{new\_t}(t_0), \mathbf{new\_b}(t_0), \mathbf{new\_n}(t_0))$ are set as the reference values that appear in Eq. 181

$$B_{11} = b_{11}(t_0) \qquad B_{12} = B_{21} = b_{12}(t_0) \qquad B_{22} = b_{22}(t_0).$$

At each time t we apply the procedure given in Section 2.5.2 to estimate the principal curvatures $\kappa_1(t)$ and $\kappa_2(t)$, and the principal directions, $\mathbf{d}_{(1)}(t)$ and $\mathbf{d}_{(2)}(t)$, at each vertex v of the deforming mesh. Eq. 103 is used to transform the estimated curvature tensors from the reference $(\mathbf{d}_{(1)}(t), \mathbf{d}_{(2)}(t), \mathbf{new\_v}(t))$ to $(\mathbf{new\_t}(t), \mathbf{new\_b}(t), \mathbf{new\_n}(t))$, namely $b_{11}(t)$, $b_{12}(t) = b_{21}(t)$ and $b_{21}(t)$.

Using the deformed reference system at time t, $(\mathbf{new\_t}(t), \mathbf{new\_b}(t), \mathbf{new\_n}(t))$, we compute the metric tensor for each vertex

$$a_{11}(t) = \mathbf{new\_t}(t)\cdot\mathbf{new\_t}(t) \quad a_{12}(t) = a_{21}(t) = \mathbf{new\_t}(t)\cdot\mathbf{new\_b}(t) \quad a_{22}(t) = \mathbf{new\_b}(t)\cdot\mathbf{new\_b}(t).$$

To get the membrane and bending strains at time t, we subtract, respectively, the coefficients of the metric and curvature tensors at t_0 from the coefficients of those metric and curvature tensors at t. It is worth remarking that our local reference systems are convected with the **reference vertices**, v_i, $v_{i,t}$ and $v_{i,b}$, whose spatial locations vary with time. Therefore, it is not necessary to carry out the coordinate transformations in the subtractions.

Finally, we re-evaluate the basis vectors of the deforming reference system for the next time instant $(\mathbf{new\_t}(t+\Delta t), \mathbf{new\_b}(t+\Delta t), \mathbf{new\_n}(t+\Delta t))$ in accordance with the new positions of $v_{i,t}$ and $v_{i,b}$, as detailed in Section 2.5.3. This procedure is repeated successively until the deformation end time is reached.

5.3.2 Surface Deformation Tensor and Surface Spin Tensor

Under the assumption that the directors coincide with the normal vectors ($\sigma_i = \kappa_{3\alpha} = 0$ in Eq. 177) and the temperature $T = 0$, we get from Eq. 178 and Eq. 179 the components of the intrinsic spin tensor and the components of the intrinsic tangential tensor in each sample point at time t

$$\begin{aligned}
M^{\alpha\beta}(t) &= [\alpha_5 A^{\alpha\beta}A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma}A^{\beta\delta} + \alpha_7 A^{\alpha\delta}A^{\beta\gamma}]\kappa_{\gamma\delta}(t) + \\
&\quad [\alpha_{10}A^{\alpha\beta}A^{\gamma\delta} + \alpha_{11}(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]\varepsilon_{\gamma\delta}(t). \\
N'^{\alpha\beta}(t) &= [\alpha_1 A^{\alpha\beta}A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]\varepsilon_{\gamma\delta}(t) + \\
&\quad [\alpha_{10}A^{\alpha\beta}A^{\gamma\delta} + \alpha_{11}(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]\kappa_{\gamma\delta}(t).
\end{aligned}$$

The components of the intrinsic deformation tensor $N^{\alpha\beta}(t)$ follow immediately from Eq. 184

$$N^{\alpha\beta}(t) = N'^{\alpha\beta}(t) - M^{\beta\gamma}(t)(a^{\alpha\delta}(t)b_{\delta\gamma}(t)). \tag{191}$$

5.3.3 Covariant Derivatives of Surface Tensors

Eq. 151 and Eq. 158 demand the covariant derivatives of the intrinsic deformation tensor $N^{\alpha\beta}$ and the intrinsic spin tensor $M^{\alpha\beta}$. Furthermore, Eq. 62 tells us that the covariant derivative of a covariant tensor differs from the usual partial derivative by a sum of linear terms of Christoffel symbols which are preceded by a negative sign. In the case of the covariant derivative of a contravariant tensor, a positive sign appears in the linear terms involving Christoffel symbols (Eq. 64). Nevertheless, both usual partial derivatives and the Christoffel symbols may be obtained numerically with use of

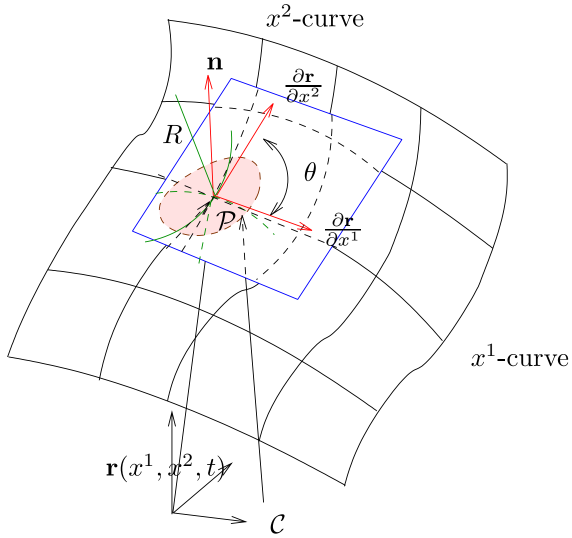

Figure 21: Geometrical quantities in a vicinity of a point $\mathcal{P}$.

finite differencing scheme as explained in Section. 2.5.5. In consequence, our problem of computing derivatives of a curvature tensor may be reduced in applying Eq. 62 on the terms calculated separately with use of the procedure given in Section 2.5.5.

One may raise the question why do not adopt the estimation procedure proposed by Rusinkiewicz, which is detailed in Section 2.5.6. This is because that we were not able to reproduce the results obtained algebraically with the procedure proposed by Rusinkiewicz.

6 A Cloth Model

We propose to model cloth as an inextensible normal-director elastic Cosserat surface, presented in Section 5.2. This choice is based on the fact that this model meets two desired requirements:

1. **"what you control is what you get" paradigm**: the applied forces and the changes in the surface's shape should be directly related, and
2. **bending representativeness**: a variety of cloth's bending behaviors should be distinguishable and reproduceable.

In the Cosserat surface the strain and bending measures are given in terms of the coefficients of the first (Eq. 138) and the second forms (Eq. 139). The first fundamental form (Eq. 9) provides us direct access to the metric measures of a surface, such as curve length, angles of tangent vectors and areas, without further reference to the ambient space, whereas the second fundamental form (Eq. 45) gives us elements to quantitate the shape of surface in the neighborhood of a point $\mathcal{P}$, or how far the surface is displaced from the tangent plane of the surface at $\mathcal{P}$.

Concerning the bending representativeness, there are two important concepts related to the shape of a surface in the vicinity of a point: the intrinsic and the extrinsic geometries. An intrinsic geometric property is the one that may be measured without leaving the surface; while an extrinsic one can only be perceived by an observer located in the ambient space. Examples for intrinsic properties of a surface are the coefficients of the first fundamental form, the surface's area, the length of a curve on the surface, and the Gaussain curvature. For exemplifying extrinsic characteristics we may mention the coefficients of the second fundamental form and the mean curvature. The intrinsic properties of most of inextensible fabrics, such as linen, cotton and jeans, are almost invariant while they deform. To distinguish the shape states, that are indistinguishable by the intrinsic properties such as buckling, we should use an extrinsic quantity. The mean curvature is an extrinsic measure and Eq. 71 says that it involves the products of the coefficients of the first and the second fundamental form. The two terms appear in the expression of the Cosserat surface's free energy (Eq. 177).

6.1 Cosserat's Formulation

We only propose to slightly modify the original formulations such that their variables have more meaningful geometric interpretation. Figure 21 illustrates the principal elements of our cloth model whose free energy has the following aspect

$$\mu\mathcal{A}(\mathbf{r},t) = \Phi^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}(t)\varepsilon_{\gamma\delta}(t) + \Psi^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta}(t)\kappa_{\gamma\delta}(t) + \Theta^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}(t)\kappa_{\gamma\delta}(t), \tag{192}$$

where

$$\begin{aligned}
\Phi^{\alpha\beta\gamma\delta} &= \frac{1}{2}[\alpha_1 A^{\alpha\beta}A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})] \\
\Psi^{\alpha\beta\gamma\delta} &= \frac{1}{2}[\alpha_5 A^{\alpha\beta}A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma}A^{\beta\delta} + \alpha_7 A^{\alpha\delta}A^{\beta\gamma}] \\
\Theta^{\alpha\beta\gamma\delta} &= [\alpha_{10} A^{\alpha\beta}A^{\gamma\delta} + \alpha_{11}(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})].
\end{aligned}$$

From several experiments we observed that these expressions may be further simplified without degrading the visual effects. We might assume that $\frac{1}{2}\alpha_1 = \frac{1}{2}\alpha_2 = \zeta$, $\frac{1}{2}\alpha_5 = \frac{1}{2}\alpha_6 = \frac{1}{2}\alpha_7 = \xi$ and $\alpha_{10} = \alpha_{11} = \phi$ and obtain the following simplification

$$\begin{aligned}
\Phi^{\alpha\beta\gamma\delta} &= \zeta[A^{\alpha\beta}A^{\gamma\delta} + (A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})] \\
\Psi^{\alpha\beta\gamma\delta} &= \xi[A^{\alpha\beta}A^{\gamma\delta} + A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma}] \\
\Theta^{\alpha\beta\gamma\delta} &= \phi[A^{\alpha\beta}A^{\gamma\delta} + (A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})].
\end{aligned} \tag{193}$$

The terms $\Phi^{\alpha\beta\gamma\delta}$ and $\Psi^{\alpha\beta\gamma\delta}$ affect predominantly the stretching and bending behaviors, respectively. Hence, we denominate ζ and ξ *elasticity coefficients* and call the parameters $\Phi^{\alpha\beta\gamma\delta}$ and $\Psi^{\alpha\beta\gamma\delta}$ *material properties*. The term $\Theta^{\alpha\beta\gamma\delta}$, in its turn, affects prevalently the number of undulations (out-of-plane behavior) on the deforming surface; thus ϕ is called the *buckling factor*.

The components of the internal surface tensors also assume more simplified version

$$M^{\alpha\beta}(t) = \Psi^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}(t) + \Theta^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}(t). \tag{194}$$

$$N'^{\alpha\beta} = \Phi^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}(t) + \Theta^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}(t), \tag{195}$$

where $\varepsilon_{\gamma\delta}$ and $\kappa_{\gamma\delta}$ are the differential variations of the components $\gamma\delta$ of the current metric and the curvature tensors with respect to the values assumed in the initial state. Having these components we may get the surface deformation tensor from Eq. 191.

Eq. 186 provides us the component $N^{3\alpha}$ of the surface deformation tensor. It requires the covariant derivatives of $M^{\alpha\gamma}$, which may be readily obtained from the procedure describes in Section. 5.3.3 if we neglect the contact director l and the director couple L,

Being available $N^{i\alpha}$, $i \in \{1,2,3\}$, at time instant t, we may take a discrete version of Eq. 187

$$\begin{aligned}
\rho\dot{v}^\beta(t) = \rho\frac{v^\beta(t+\Delta t) - v^\beta(t)}{\Delta t} &= N^{\beta\alpha}_{|\alpha}(t) - b^\beta_\gamma(t)M^{\gamma\alpha}_{|\alpha}(t) + \rho F^\beta(t) - \rho b^\beta_\gamma(t)\bar{L}^\gamma(t) \\
&\approx N^{\beta\alpha}_{|\alpha}(t) - b^\beta_\gamma(t)M^{\gamma\alpha}_{|\alpha}(t) + \rho F^\beta(t); \\
\rho\dot{v}^3(t) = \rho\frac{v^3(t+\Delta t) - v^3(t)}{\Delta t} &= M^{\beta\alpha}_{|\alpha\beta}(t) + b_{\alpha\beta}(t)N^{\beta\alpha}_{|\alpha}(t) + \rho F^3(t) + (\rho\bar{L}^\gamma)_{|\beta}(t) \\
&\approx M^{\beta\alpha}_{|\alpha\beta}(t) + b_{\alpha\beta}(t)N^{\beta\alpha}_{|\alpha}(t) + \rho F^3(t),
\end{aligned} \tag{196}$$

from which we may compute the velocity $\mathbf{v} = (v^1, v^2, v^3)$ at time $t + \Delta t$

$$\begin{aligned}
\rho v^\beta(t+\Delta t) &= \Delta t(N^{\beta\alpha}_{|\alpha}(t) - b^\beta_\gamma(t)M^{\gamma\alpha}_{|\alpha}(t) + \rho F^\beta(t)) + \rho v^\beta(t); \\
\rho v^3(t+\Delta t) &= \Delta t(M^{\beta\alpha}_{|\alpha\beta}(t) + b_{\alpha\beta}(t)N^{\beta\alpha}_{|\alpha}(t) + \rho F^3(t)) + \rho v^3(t).
\end{aligned}$$

Then, the position vector $\mathbf{r}(t) = (x(t), y(t), z(t))$ follows immediately

$$\mathbf{r}(t+\Delta t) = \mathbf{r}(t) + \mathbf{v}(t+\Delta t)\Delta t. \tag{197}$$

The repeated covariant differentiation of $M^{\beta\alpha}$ is obtained by differentiating $M^{\beta\alpha}_{|\beta}$ and is a special case of Eq. 66

$$M^{\beta\alpha}_{|\alpha\beta} = \frac{\partial M^{\beta\alpha}_{|\alpha}}{\partial u^\beta} + \Gamma^\beta_{1\beta}M^{1\alpha}_{|\alpha} + \Gamma^\beta_{2\beta}M^{2\alpha}_{|\alpha} + \Gamma^\alpha_{1\beta}M^{\beta 1}_{|\alpha} + \Gamma^\alpha_{2\beta}M^{\beta 2}_{|\alpha} - \Gamma^1_{\alpha\beta}M^{\beta\alpha}_{|1} - \Gamma^2_{\alpha\beta}M^{\beta\alpha}_{|2}, \tag{198}$$

where $M^{\beta\alpha}_{|\beta}$ may be computed from $M^{\beta\alpha}$ with use of the discrete form of Eq. 64.

It is worth remarking that all physical quantities should be given with respect to the convected coordinates (Section 4). The position-vector $\mathbf{r}_{wc}(t)$ with respect the cloth reference system is actually

$$\mathbf{r}_{wc}(t) = x(t)\mathbf{new\_t}(t) + y(t)\mathbf{new\_b}(t) + z(t)\mathbf{new\_n}(t).$$

It is, however, easier and more intuitive to specify the external forces $\mathbf{F} = (F^1, F^2, F^3)$ in the cloth reference system. Hence, we may alternatively take a discrete version of Eq. 150

$$\begin{aligned} \rho\frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} &= \mathbf{N}^1_{|1} + \mathbf{N}^2_{|2} + \rho\mathbf{F} \\ \rho\mathbf{v}(t+\Delta t) &= \Delta t(\mathbf{N}^1_{|1} + \mathbf{N}^2_{|2} + \rho\mathbf{F}) + \rho\mathbf{v}(t) \end{aligned} \tag{199}$$

In this case, we should determine the covariant derivatives $\mathbf{N}^1_{|1}$ and $\mathbf{N}^2_{|2}$. From Eq. 148, i.e.

$$\begin{aligned} \mathbf{N}^1 &= N^{11}\mathbf{new\_t}(t) + N^{21}\mathbf{new\_b}(t) + N^{31}\mathbf{new\_n}(t) \\ \mathbf{N}^2 &= N^{12}\mathbf{new\_t}(t) + N^{22}\mathbf{new\_b}(t) + N^{32}\mathbf{new\_n}(t), \end{aligned}$$

and applying Eq. 79, Eq. 81 and Eq. 186, we get the following expressions that provide us $\mathbf{N}^1_{|1}$ and $\mathbf{N}^2_{|2}$ in the cloth reference system

$$\begin{aligned} \mathbf{N}^1_{|1} &= (N^{11}\mathbf{new\_t}(t))_{|1} + (N^{21}\mathbf{new\_b}(t))_{|1} + (N^{31}\mathbf{new\_n}(t))_{|1} \\ &= N^{11}_{|1}(\mathbf{new\_t}(t)) + N^{11}(\mathbf{new\_t}(t))_{|1} + N^{21}_{|1}(\mathbf{new\_b}(t)) + N^{21}(\mathbf{new\_b}(t))_{|1} \\ &\quad + N^{31}_{|1}(\mathbf{new\_n}(t)) + N^{31}(\mathbf{new\_n}(t))_{|1} \\ &= N^{11}_{|1}(\mathbf{new\_t}(t)) + N^{11}(b_{11}\mathbf{new\_n}(t)) + N^{21}_{|1}(\mathbf{new\_b}(t)) + N^{21}(b_{21}\mathbf{new\_n}(t)) \\ &\quad + N^{31}_{|1}(\mathbf{new\_n}(t)) - N^{31}(b^1_1\mathbf{new\_t}(t) + b^2_1\mathbf{new\_b}(t)) \\ &= (N^{11}_{|1} - N^{31}b^1_1)\mathbf{new\_t}(t) + (N^{21}_{|1} - N^{31}b^2_1)\mathbf{new\_b}(t) \\ &\quad + (N^{11}b_{11} + N^{21}b_{21} + N^{31}_{|1})\mathbf{new\_n}(t) \\ &= (N^{11}_{|1} - N^{31}b^1_1)\mathbf{new\_t}(t) + (N^{21}_{|1} - N^{31}b^2_1)\mathbf{new\_b}(t) \\ &\quad + (N^{11}b_{11} + N^{21}b_{21} + (M^{11}_{|11} + M^{12}_{|21}))\mathbf{new\_n}(t) \\ \mathbf{N}^2_{|2} &= (N^{12}\mathbf{new\_t}(t))_{|2} + (N^{22}\mathbf{new\_b}(t))_{|2} + (N^{32}\mathbf{new\_n}(t))_{|2} \\ &= N^{12}_{|2}(\mathbf{new\_t}(t)) + N^{12}(\mathbf{new\_t}(t))_{|2} + N^{22}_{|2}(\mathbf{new\_b}(t)) + N^{22}(\mathbf{new\_b}(t))_{|2} \\ &\quad + N^{32}_{|2}(\mathbf{new\_n}(t)) + N^{32}(\mathbf{new\_n}(t))_{|2} \\ &= N^{12}_{|2}(\mathbf{new\_t}(t)) + N^{12}(b_{12}\mathbf{new\_n}(t)) + N^{22}_{|2}(\mathbf{new\_b}(t)) + N^{22}(b_{22}\mathbf{new\_n}(t)) \\ &\quad + N^{32}_{|2}(\mathbf{new\_n}(t)) - N^{32}(b^1_2\mathbf{new\_t}(t) + b^2_2\mathbf{new\_b}(t)) \\ &= (N^{12}_{|2} - N^{32}b^1_2)\mathbf{new\_t}(t) + (N^{22}_{|2} - N^{32}b^2_2)\mathbf{new\_b}(t) \\ &\quad + (N^{12}b_{12} + N^{22}b_{22} + N^{32}_{|2})\mathbf{new\_n}(t) \\ &= (N^{12}_{|2} - N^{32}b^1_2)\mathbf{new\_t}(t) + (N^{22}_{|2} - N^{32}b^2_2)\mathbf{new\_b}(t) \\ &\quad + (N^{12}b_{12} + N^{22}b_{22} + (M^{21}_{|12} + M^{22}_{|22}))\mathbf{new\_n}(t) \end{aligned} \tag{200}$$

Now, our task is reduced to the estimation of the covariant derivatives. This may be performed with use of a discrete form of Eq. 64 or Eq. 198.

Figure 22 presents the iterative simulation data flow. Starting from a sampled surface, we compute the components of the internal forces $N^{\beta\alpha}$, $M^{\gamma\alpha}$ and their derivatives for each sample at a time instant t. Then, we calculate the forces $\mathbf{N}^1_{|1}$ and $\mathbf{N}^2_{|2}$ with use of Eq. 200. Substituting them in Eq. 199, we get the sample point's acceleration $\dot{\mathbf{v}}(t)$ and velocity $\mathbf{v}(t+\Delta t)$. This allows us to obtain the sample position at next time instant $t + \Delta t$. Successively, we get a series of mesh evolving forward in time under physical and geometrical constraints. We remark that under the assumption that the director vector is undeformed, we have neglected Eq. 188 in our proposal.

6.2 Boundary Conditions

Boundary conditions for the cloth simulation as a Cosserat mesh are fundamental for simulation stability.

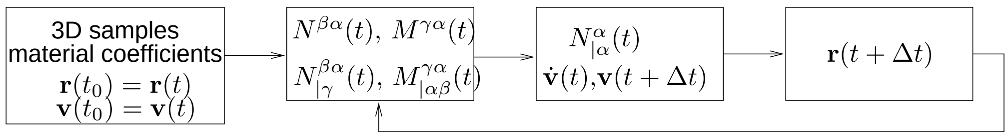

Figure 22: Simulation data flow.

Lattice Boltzmann models do not directly simulate the evolution of the flow velocity. Instead, they calculate the dynamics of particle populations which stem from a microscopic description of the fluid. While the macro- scopic pressure and velocity fields are easily calculated from the particle populations, the reverse procedure is more contrived. Thus, implementing a velocity condi- tion on straight boundaries boils down to finding a way to translate from macroscopic flow variables to particle populations. This problem has been approached by au- thors from different viewpoints, some of them based on the kinetic theory of gases, and some of them on a hy- drodynamic description of fluids. Although the numerical scheme of the LB method is derived from microscopic physics, it is able to recover ac- curate solutions of the macroscopic Navier-Stokes equa- tions. This can be shown in various ways through an asymptotic analysis, in which particle populations are formally related to macroscopic flow variables. An anal- ysis of this type is however not always conducted in the literature for boundary conditions, and only little is known about their hydrodynamic limit. In the present article, the boundary conditions are therefore inspected with help of a Chapman-Enskog multi-scale analysis

7 Conclusions

The main motivation of our work is to devise a cloth model that has an intuitive interface and produces realistic cloth's appearance. We conjecture that the Cosserat surface is a potential candidate. In this report we provide the mathematical foundation of the Cosserat surface. Nevertheless, as mentioned in Section 2.5, we are working on surface samples without the knowledge of its parametrization. Hence, as a contribution, we develop in this report a series of discrete formulations that allow the estimation of the geometric and physical quantities from a mesh of arbitrary topology. Moreover, we present a procedure to implement it in a computational framework.

As further work we will implement the proposed procedure and analyze the valid value range for the elasticity coefficients and the buckling factor.

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References

[1] J. Amirbayat and J.W.S. Hearle. The anatomy of buckling of textile fabrics: Drape and conformability. *Journal iof the Textile Institute*, 80(1):51–70, 1989.

[2] David Baraff and Andrew Witkin. Large steps in cloth simulation. In *Proceedings of SIGGRAPH '98*, pages 43–54, 1998.

[3] Harlen Costa Batagelo. *A GPU-based architecture for supporting 3D interactions*. PhD thesis, School of Computer and Electrical Engineering, State University of Campinas, 2007. (in Portuguese).

[4] M. Berger and B. Gostiaux. *Differential Geometry: Manifolds, Curves, and Surfaces*. Springer-Verlag, New York, 1988.

[5] Miklos Bergou, Max Wardetzky, David Harmon, Denis Zorin, and Eitan Grinspun. A quadratic bending model for inextensible surfaces. In *Proceedings of the Fourth Eurographics Symposium*

on Geometry Processing, SGP '06, pages 227–230, Aire-la-Ville, Switzerland, Switzerland, 2006. Eurographics Association.

[6] D. E. Breen, D. H. House, and P. H. Getto. A physically-based particle model of woven cloth. *The Visual Computer*, 8:264–277, 1992.

[7] D. E. Breen, D. H. House, and M. J. Wozny. Predicting the drape of woven cloth using interacting particles. *Proceedings of SIGGRAPH 1994*, page ???, 1994.

[8] R. Bridson, S. Marino, and R. Fedkiw. Simulation of clothing with folds and wrinkles. In *Eurographics/SIGGRAPH Symposium on Computer Animation*, 2003.

[9] M. P. Carmo. *Differential geometry of curves and surfaces.* Prentice Hall Inc., New Jersey, 1st. edition, 1976.

[10] Bijian Chen and Muthu Govindaraj. A physically based model of fabric drape using flexible shell theory. *Textile Research Journal*, 65(6):324–330, 1995.

[11] Kwang-Jin Choi and Hyeong-Seok Ko. Stable but responsive cloth. *Proceedings of SIGGRAPH 2002*, pages 604–611, 2002.

[12] Vanio Fragoso de Melo. *Modeling and Control of Drappings and Folding in Deformable Surfaces.* PhD thesis, University of Campinas, 2004. (in Portuguese: Modelagem e Controle de Caimento e Dobras em SuperfÃcies DeformÃ¡veis.

[13] Leandro de Pinho Monteiro. Numerical solutions for a cloth model based on the cosserat surface approach. Master's thesis, University of Campinas, 2008. In Portuguese: Soluções Numéricas em um Modelo de Tecidos Baseado na SuperfÃcie de Cosserat.

[14] G. Debunne, M. Desbrun, A. Barr, and M.P. Cani. Interactive multiresolution animation of deformable models. In Springer-Verlag, editor, *Proceedings Workshop on Computer Animation and Simulation '99*, 1999.

[15] Alan H. Barr Demetri Terzopoulos, John C. Platt and Kurt Fleischer. Elastically deformable models. *Computer Graphics*, 21(4):205–214, July 1987.

[16] B. Eberhardt, A. Weber, and W. Strasser. A fast, flexible, particle-system model for cloth draping. *IEEE Computer Graphics and Application*, 16(5):52–59, 1996.

[17] J.W. Eischen, S. Deng, and T.G. Clapp. Finite-element modeling and control of flexible fabric parts. *IEEE Computer Graphics and Application*, 16(5):71–S80, 1996.

[18] J. L. Ericksen and C. Truesdell. Exact theory of stress and strain in rods and shells. *Archive for Rational Mechanics and Analysis*, 1(1):295–323, jan 1957.

[19] C. Feynman. Modeling the appearance of cloth. Master's thesis, Massachusets Institute of Technology, Massachusets, 1986.

[20] Wilhelm Flügge. *Tensor Analysis and Continuum Mechanics.* Springer Berlin Heidelberg, 1972.

[21] Henri Gouraud. *Computer Display of Curved Surfaces.* PhD thesis, 1971. AAI7127878.

[22] Jens Gravesen and Michael Ungstrup. Constructing invariant fairness measures for surfaces. *Advances in Computational Mathematics*, 17:67–88, 2002.

[23] A. E. Green, P.M. Naghdi, and W.L. Wainwright. A general theory of a cosserat surface. *Arch. Rat. Mech. An.*, 20:287–308, 1965.

[24] Eitan Grinspun, Anil Hirani, Mathieu Desbrun, and Peter Schröder. Discrete shells. *Eurographics/SIGGRAPH Symposium on Computer Animation*, pages 62–67, 2003.

[25] Jinlian Hu. *Structure and mechanics of woven fabrics.* Woodhead Publishing Limited, England, 1st edition, 2004.

[26] Feng Ji, Ruqin Li, and Yiping Qiu. Simulate the dynamic draping behavior of woven and knitted fabrics. *Journal of Industrial Textiles*, 35(3):201–215, January 2006.

[27] Shuangshuang Jin, Robert R. Lewis, and David West. A comparison of algorithms for vertex normal computation. *The Visual Computer*, 21(1-2):71–82, 2005.

[28] Young-Min Kang, Jeong-Hyeon Choi, Hwan-Gue Cho, and Do-Hoon Lee. An efficient animation of wrinkled cloth with approximate implicit integration. *The Visual Computer*, 17(3):147–157, 2001.

[29] Erwin Kreyszig. *Differential Geometry.* Dover Publications, New York, 1991.

[30] David Lovelock and Hanno Rund. *Tensors, Differential Forms, and Variational Principles*. Dover Publications, Inc., 1989.

[31] Nelson Max. Weights for computing vertex normals from facet normals. *Journal of Graphics Tools*, pages 1–6, 1999.

[32] Mark Meyer, Mathieu Desbrun, Peter Schröder, and Alan H. Barr. Discrete differential-geometry operators for triangulated 2-manifolds. In Hans-Christian Hege and Konrad Polthier, editors, *Visualization and Mathematics III: Geometry, Topology and Numerics*. Springer Verlag Heidelberg, 2003.

[33] Nadia M. Thalmann Michel Carignan, Ying Yang and Daniel Thalmann. Dressing animated synthetic actors with complex deformable clothes. *Computer Graphics*, 26(2):99–104, July 1992.

[34] H. Mollmann. *Introduction to the Theory of Thin Shells*. John Wiley & Sons, Inc., 1981. ISBN 0471280569.

[35] Cyril Ngo Ngoc. Nonlinear cloth simulation. Technical report, Institut National de Recherche en Informatique et en Automatique, 2004.

[36] William H. Press, Brian P. Flannery, and Saul A. Teukolskyand William T. Vetterling. *Numerical Recipes in C: the Art of Scientific Computing*. Cambridge University Press, USA, 2nd edition, 1992. ISBN 0-521-43108-5.

[37] Xavier Provot. Deformation constraints in a mass-spring model to describe rigid cloth behavior. In *Proceedings of Graphics Interface*, pages 147–154, 1995.

[38] Szymon Rusinkiewicz. trimesh. `http://gfx.cs.princeton.edu/proj/trimesh2/`. accessed in March 2018.

[39] Szymon Rusinkiewicz. Estimating curvatures and their derivatives on triangle meshes. In *3DPVT '04: Proceedings of the 3D Data Processing, Visualization, and Transmission, 2nd International Symposium on (3DPVT'04)*, pages 486–493, Washington, DC, USA, 2004. IEEE Computer Society.

[40] J. C. Simo and D. D. Fox. On a stress resultant geometrically exact shell model. part i: Formulation and optimal parameterization. *Computer Methods in Applied Mechanics and Engeneering*, 72:267–304, 1989.

[41] J. C. Simo and D. D. Fox. On a stress resultant geometrically exact shell model. part ii: The linear theory; computational aspects. *Computer Methods in Applied Mechanics and Engeneering*, 73:53–92, 1989.

[42] J. C. Simo and D. D. Fox. On a stress resultant geometrically exact shell model. part iii: Aspects of the nonlinear theory. *Computer Methods in Applied Mechanics and Engeneering*, 79:21–70, 1989.

[43] D. J. Struik. *Lectures on Classical Differential Geometry*. Dover Publications, Inc., 2nd. edition, 1961.

[44] Bernhard Thomaszewski, Markus Wacker, and Wolfgang Straßer. A consistent bending model for cloth simulation with corotational subdivision finite elements. In *Proceedings of the 2006 ACM SIGGRAPH/Eurographics Symposium on Computer Animation*, SCA '06, pages 107–116, Aire-la-Ville, Switzerland, Switzerland, 2006. Eurographics Association.

[45] Pascal Volino, Martin Courchesne, and Nadia Magnenat Thalmann. Versatile and efficient techniques for simulating cloth and other deformable objects. In *Proceedings of Siggraph 1995*, pages 137–144, 1995.

[46] Biao Wang, Zhuang Wu, Qingping Sun, and Matthew M.F. Yuen. A deformation model of thin flexible surface. In *Proceedings of The Sixth International Conference in Central Europe on Computer Graphics and Visualization'98*, pages 440–447, 1998.

[47] Martin Wicke, Denis Steinemann, and Markus Gross. Efficient animation of point-sampled thin shells. *Computer Graphics Forum*, 24(3):667–676, sep 2005.